# Boolean Algebra Revisited with K-maps

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Updated February 7, 2025

Boolean algebra is not easy, and some steps get counterintuitive. Karnaugh maps provide a systematic way to guide you through these manipulations.
Here's a problem from the Fall 2024 Midterm 1:

> Use **algebraic manipulation** to reduce the following expression to two literals
>
> $$f = A + \overline{B}(\overline{A} + C)$$

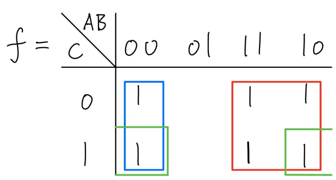
A natural start is to distribute

$$f = A + \overline{A}\,\overline{B} + \overline{B}C$$

and get stuck with the letters. Let's try another way: first find the answer using K-map, and translate the steps back to boolean algebra.
As you'll see, this may not lead to the simplest derivation, but it provides a very intuitive sequence of logic that you can easily follow.

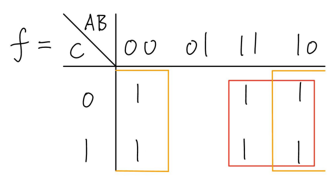
## 1 K-map Solution

First, use K-map to obtain the simplest Sum of Products (SoP) expression.




$f = A + \overline{A}\,\overline{B} + \overline{B}C$

$\Longrightarrow$




$f = A + \overline{B}$

## 2 From K-map to Algebra

We took two steps in the K-map above. Let's translate each into boolean algebra.

1. Eliminate $\overline{B}C$.
2. Expand $\overline{A}\,\overline{B}$ to $\overline{B}$.

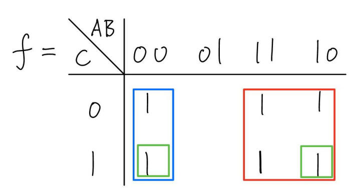
### 2.1 Eliminating $\overline{B}C$

$\overline{B}C$ is redundant suggests that all of $\overline{B}C$'s constituent minterms can be absorbed by other terms.

$$\overline{B}C = A\overline{B}C + \overline{A}\,\overline{B}C$$

As you can see,

- $A\overline{B}C$ can be absorbed by $A$;
- $\overline{A}\,\overline{B}C$ can be absorbed by $\overline{A}\,\overline{B}$.



Let's draft a skeleton:

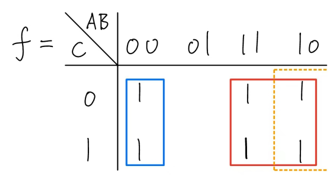
$$
\begin{aligned}
f &= A + \overline{A}\,\overline{B} + \overline{B}C \\
&= A + \overline{A}\,\overline{B} + A\overline{B}C + \overline{A}\,\overline{B}C && \text{(Split to minterm)} \\
&= (A + A\overline{B}C) + (\overline{A}\,\overline{B} + \overline{A}\,\overline{B}C) && \text{(Regroup)} \\
&= A + \overline{A}\,\overline{B} && \text{(Absorption)}
\end{aligned}
$$

With the skeleton, it's simple to fill out the intermediate steps:

$$
\begin{aligned}
f &= A + \overline{A}\,\overline{B} + \overline{B}C \\
&= A + \overline{A}\,\overline{B} + (1)\overline{B}C \\
&= A + \overline{A}\,\overline{B} + (A + \overline{A})\overline{B}C \\
&= A + \overline{A}\,\overline{B} + A\overline{B}C + \overline{A}\,\overline{B}C && \text{(Split to minterm)} \\
&= (A + A\overline{B}C) + (\overline{A}\,\overline{B} + \overline{A}\,\overline{B}C) && \text{(Regroup)} \\
&= A(1 + \overline{B}C) + \overline{A}\,\overline{B}(1 + C) \\
&= A(1) + \overline{A}\,\overline{B}(1) \\
&= A + \overline{A}\,\overline{B} && \text{(Absorption)}
\end{aligned}
$$

### 2.2 Expanding $\overline{A}\,\overline{B}$ to $\overline{B}$

Expanding a K-map box means the newly introduced terms are already covered by existing terms. We can "extract" the added portion and merge it with the new box.




$$
\begin{aligned}
f &= A + \overline{A}\,\overline{B} \\
&= (A + A\overline{B}) + \overline{A}\,\overline{B} && \text{(Extract from } A\text{)} \\
&= A + (A\overline{B} + \overline{A}\,\overline{B}) && \text{(Regroup)} \\
&= A + \overline{B} && \text{(Absorption)}
\end{aligned}
$$

Of course, some intermediate steps are needed—work through them as an exercise.
K-map boxes represent SoP expressions. However, a simpler derivation can be done using PoS (Product of Sums): $f = A + \overline{A}\,\overline{B} = (A + \overline{A})(A + \overline{B}) = A + \overline{B}$.

## 3 Brute Force Translation

If the K-map transformations are not obvious, you can always break expressions down into a sum of minterms as a last resort. Duplicate and group like terms, and then merge them back into the final K-map expression.

$$
\begin{aligned}
f &= A + \overline{A}\,\overline{B} + \overline{B}C \\
&= ABC + A\overline{B}C + AB\overline{C} + A\overline{B}\,\overline{C} + \overline{A}\,\overline{B}C + \overline{A}\,\overline{B}\,\overline{C} + A\overline{B}C + \overline{A}\,\overline{B}C \\
&= ABC + A\overline{B}C + AB\overline{C} + A\overline{B}\,\overline{C} + \overline{A}\,\overline{B}C + \overline{A}\,\overline{B}\,\overline{C} + A\overline{B}C + A\overline{B}\,\overline{C} \\
&= A + \overline{B}
\end{aligned}
$$

This will generate a lot of intermediate terms, but the fundamental logic remains intact.

# 4 Summary

Direct algebraic simplification can be challenging, but reverse engineering with K-maps provides a systematic graphical approach:

1. First, simplify with the distributive property and DeMorgan's law.
2. Use a K-map to find the final minimal form.
3. Break down the K-map simplification into distinct steps.
4. For each step,
   (a) Establish a skeleton: identify which variables should be split or absorbed in the K-map.
   (b) Write out the skeleton of algebraic expressions.
   (c) Flesh out the steps using basic axioms.
5. If you can't easily relate the K-map transformations, you can always reduce the expressions to minterms and work with those instead.

We start big with the final simplified expression, analyze how the K-map achieves it step by step, and then refine each transformation into its algebraic form.

Written by Ming Gong in Spring 2025.