Blgebra

$\sin n\pi = 0$
$1 - \cos n\pi = 2$ for odd n
$\arctan(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}$, $\arctan(\sqrt{3}) = \frac{\pi}{3}$

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$
$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$
$\cos x \cos y = \frac{1}{2}[\cos(x-y) + \cos(x+y)]$
$\sin x \cos y = \frac{1}{2}[\sin(x-y) + \sin(x+y)]$

$\sin a \pm \sin b = 2 \sin \frac{a+b}{2} \cos \frac{a \mp b}{2}$
$\cos a + \cos b = 2 \cos \frac{a+b}{2} \cos \frac{a-b}{2}$
$\cos a - \cos b = -2 \sin \frac{a+b}{2} \sin \frac{a-b}{2}$

$C\cos(\omega_0 t + \theta) = C\cos(\theta)\cos(\omega_0 t) - C\sin(\theta)\sin(\omega_0 t)$
$C\sin(\omega_0 t + \theta) = C\sin(\theta)\cos(\omega_0 t) + C\cos(\theta)\sin(\omega_0 t)$

$\theta = \tan^{-1}(-\frac{b}{a})$, $\pm\pi$ when $a < 0$
$\sin t = \cos(t - \frac{\pi}{2})$
$-\cos t = \sin(t - \frac{\pi}{2})$

$\cos x = \frac{1}{2}[e^{jx} + e^{-jx}]$
$\sin x = \frac{1}{2j}[e^{jx} - e^{-jx}]$
$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$

$z^* = a - jb = re^{-j\theta}$
$u^* v^* = (uv)^*$

$\angle z = \tan^{-1}(\frac{b}{a})$, $\pm\pi$ in Q2 and Q3

$z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{j\frac{\theta + 2\pi m}{n}}$

$(s+a)(s+b)(s+c) = s^3 + (a+b+c)s^2 + (ab+bc+ca)s + abc$

Integrals

$\int \cos^2 at\, dt = \frac{t}{2} + \frac{\sin 2at}{4a}$

$\int t \cos at\, dt = \frac{1}{a^2}(\cos at + at \sin at)$
$\int t \sin at\, dt = \frac{1}{a^2}(\sin at - at \cos at)$
$\int t^2 \cos at\, dt = \frac{1}{a^3}(2at \cos at - 2 \sin at + a^2t^2 \sin at)$
$\int t^2 \sin at\, dt = \frac{1}{a^3}(2at \sin at + 2 \cos at - a^2t^2 \cos at)$

$\int te^{at}\, dt = \frac{1}{a^2} e^{at}(at - 1)$
$\int t^2 e^{at}\, dt = \frac{1}{a^3} e^{at}(a^2t^2 - 2at + 2)$

$\int e^{at} \cos bt\, dt = \frac{1}{a^2+b^2} e^{at}(a \cos bt + b \sin bt)$
$\int e^{at} \sin bt\, dt = \frac{1}{a^2+b^2} e^{at}(a \sin bt - b \cos bt)$

$\int \frac{1}{x^2+a^2}\, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}$

Signals

$\mathcal{E}_f = \int_{-\infty}^{\infty} |f(t)|^2 dt$ (complex);
$P_f = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 dt$;
rms power $= \sqrt{P_f}$

Cont; analog; periodic (extension); (non/anti)causal; energy/power (both); deterministic/stochastic (info)

$\int f(t) \cdot \delta(t - t_0) dt = f(t_0)$ (f continuous at t_0)

$f(2x-6)$: shift by 6, scale by 2;
$f(2(x-6))$: scale by 2, shift by 6

$f_e(t) = \frac{1}{2}[f(t) + f(-t)]$
$f_o(t) = \frac{1}{2}[f(t) - f(-t)]$

Systems

$\mathcal{T}$: $\sum_{k=0} a_k D^k y(t) = \sum_{l=0} b_l D^l f(t)$

Linear $\mathcal{T}[kf_1(t) + f_2(t)] = ky_1(t) + y_2(t)$.
Lin if a_k, b_l are not functions of $y(t), f(t)$
E. $\sin \dot{y}(t) + t^2 y(t) = (t+3)f(t)$

Time-inv $\mathcal{T}[f(t-\tau)] = y(t-\tau)$.
a_k, b_l indep of t (const coeff)
Let $g(t) \equiv f(t-\tau)$, find $z(t) = \mathcal{T}[g(t)]$, cmp $y(t-\tau)$

Causal $y(t)$ dep only on $f(\tau)$, $\tau \le t$. Compare t and τ.

Instantaneous y only dep f at present (no $\int$, no memory)

Invertible given $y(t)$, we can know $f(t)$ (ideal diff is not)

Conv prop

$c(t) \equiv \int_{-\infty}^{\infty} f(\tau) g(t-\tau)\, d\tau$
$c[n] \equiv \sum_{m=-\infty}^{\infty} f[m] g[n-m]$

$f * g = g * f$
$f * (g + h) = f * h + g * h$
$f * (g * h) = (f * g) * h$
pf: $f * (g * h) = f * (h * g)$
$= \int f(\tau_1) \int h(\tau_2)\, g(t - \tau_2 - \tau_1)\, d\tau_2\, d\tau_1$
$= \int h(\tau_2) \int f(\tau_1)\, g(t - \tau_1 - \tau_2)\, d\tau_1\, d\tau_2$

$f(t - T_1) * g(t - T_2) = c(t - T_1 - T_2)$

$f(at) * g(at) = |\frac{1}{a}|\, c(at)$ (even/odd)

$f^{(m)}(t) * g^{(n)}(t) = c^{(m+n)}(t)$
pf: $\dot{f}(\tau) = \lim_{T\to 0} f(\tau) - f(\tau - T)$

Graph: shift left by $+t$, and reflect;
Every τ replaced by $t - \tau$; Reverted

Conv table

$f(t) * \delta(t - T) = f(t - T)$

$u(t) * u(t) = t\, u(t)$

$e^{at}\, u(t) * u(t) = \frac{1 - e^{at}}{-a}\, u(t)$

$e^{at}\, u(t) * e^{bt}\, u(t) = \frac{e^{at} - e^{bt}}{a - b}\, u(t)$ $\quad a = b$, $te^{at}\, u(t)$

$e^{at}\, u(t) * e^{bt}\, u(-t) = \frac{e^{at}\, u(t) + e^{bt}\, u(-t)}{b - a}$ $\quad \Re(b) > \Re(a)$

$te^{at}\, u(t) * e^{at}\, u(t) = \frac{1}{2}\, t^2 e^{at}\, u(t)$

$t^m\, u(t) * t^n\, u(t) = \frac{m!\, n!}{(m+n+1)!}\, t^{m+n+1}\, u(t)$

Don't forget $[u(t + T_1) - u(t - T_2)]$ term

LTI response

$Q(D)y(t) = P(D)f(t)$, typically integrating f
Assume causal input $f(t)u(t)$

$y_{zs}(t) = f(t) * h(t)$ from input
$y_{zs}(0^-) = 0$, $y_{zs}(0^+) \ne 0$

Let $h(t) = \mathcal{T}[\delta(t)]$ (impulse response)
$y_{zs}(t) = \mathcal{T}[f(t)] = \mathcal{T}[f(t) * \delta(t)]$
$= \mathcal{T}[\lim \sum f(n\Delta\tau)\delta(t - n\Delta\tau)\Delta\tau]$
$= \lim \sum f(n\Delta\tau) h(t - n\Delta\tau)\Delta\tau = f * h$

$y_{zi}(t)$ from ini, $f(t) = 0$, $Qy_{zi}(t) = 0$
$y_{zi}(0^-) = y_{zi}(0^+)$, $\dot{y_{zi}}(0^-) = \dot{y_{zi}}(0^+)$

Ortho set

$\mathcal{E}_e = \int_{t_1}^{t_2} [e(t)]^2 dt$
$= \int_{t_1}^{t_2} f^2(t) dt - 2 \sum c_i \int_{t_1}^{t_2} f(t) x_i(t) dt + \int_{t_1}^{t_2} (\sum c_i x_i(t))^2 dt$
$= \mathcal{E}_f - 2 \sum_{c_i} \langle f, x_i \rangle$
$+ (\sum c_i^2 \int_{t_1}^{t_2} x_i(t)^2 dt + \sum_{i \ne j} c_i c_j \int_{t_1}^{t_2} x_i(t) x_j(t) dt)$

$\frac{\partial \mathcal{E}_e}{\partial c_i} = 0 = -2\langle f(t), x_i(t) \rangle + 2\mathcal{E}_i c_i$
$\mathcal{E}_e^{\min} = \mathcal{E}_f - \sum_{i=1}^{N} c_i^2 \mathcal{E}_i$

$c_i = \frac{1}{\mathcal{E}_i} \langle f, x_i \rangle = \frac{\int f(t) x(t) dt}{\int x_i^2(t) dt}$

For ortho, $E_z = E_x + E_y$

$|u + v|^2 = |u|^2 + |v|^2 + u^* v + v^* u$

$\langle x(t), y(t) \rangle = \int_{t_1}^{t_2} x(t) y(t)^*\, dt = \int_{t_1}^{t_2} x(t) y(t) dt$ if real
Use prod $\to$ sum identities

FS

$a_0 = \frac{1}{T_0}\int_{T_0} f(t)\,dt$
$a_n = \frac{2}{T_0}\int_{T_0} f(t)\cos(n\omega_0 t)\,dt$
$b_n = \frac{2}{T_0}\int_{T_0} f(t)\sin(n\omega_0 t)\,dt$
Energy: T_0 for $n=0$; $T_0/2$ else

Half wave sym $f(t-\frac{T_0}{2}) = -f(t)$
$a_{n_{\text{odd}}} = \frac{4}{T_0}\int_0^{T_0/2} f(t)\cos(n\omega_0 t)\,dt$

$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$
$F_n = \frac{1}{T_0}\int_{T_0} f(t)e^{-jn\omega_0 t}\,dt$

$C_n\cos(n\omega_0 t+\theta_n) = \frac{C_n}{2}\left(e^{j(n\omega_0 t+\theta_n)} + e^{-j(n\omega_0 t+\theta_n)}\right)$
$= (\frac{C_n}{2}e^{j\theta_n})e^{jn\omega_0 t} + (\frac{C_n}{2}e^{-j\theta_n})e^{-jn\omega_0 t}$

$F_n = \frac{C_n}{2}e^{j\theta_n} = \frac{1}{2}(a_n - jb_n) = |F_n|e^{j\angle F_n}$
$F_{-n} = \frac{C_n}{2}e^{-j\theta_n} = \frac{1}{2}(a_n + jb_n)$

Existence

Weak: finite $\int$, fin bounds a, b, fin power
Strong: fin min/max/discont over T_0, $\to \frac{f(t_0^+)+f(t_0^-)}{2}$

FS prop

Time shift $f(t-t_0) \to F_n e^{-jn(\omega_0 t_0)}$
$|F_n|$ same; $\angle F_n$ shifted by $-(\omega_0 t_0)n$

Reversal $f(-t) \to F_{-n}$
Scaling $T = \frac{T_0}{a}$, $\omega = a\omega_0$

Multiplication (same T_0): $f(t)g(t) \to F_n * G_n$
$\frac{1}{T_0}\int_{T_0} f(t)g(t)e^{jn\omega_0 t}\,dt$
$= \frac{1}{T_0}\int(\sum F_m e^{jm\omega_0 t})(\sum G_k e^{jk\omega_0 t})e^{-jn\omega_0 t}\,dt$
$= \sum_m\sum_k F_m G_k \frac{1}{T_0}\int_{T_0} e^{j(m+k-n)\omega_0 t}\,dt$
$= \sum_m\sum_k F_m G_k \langle e^{j(m+k)\omega_0 t}, e^{jn\omega_0 t}\rangle$
$= \sum_{k=-\infty}^{\infty} G_k F_{n-k}$

Conjugation $f(t)^* = F_{-n}^*$

Parseval (power sig): $P_f = \frac{1}{T_0}\int_{T_0} f(t)f(t)^* dt$
$= \frac{1}{T_0}\int_{T_0}(\sum_n F_n e^{jn\omega_0 t})(\sum_m F_m e^{jm\omega_0 t})^* dt$
$= \sum_n\sum_m F_n F_m^* \frac{1}{T_0}\int_{T_0} e^{j(n-m)\omega_0 t}dt$
$= \sum_n |F_n|^2 \cdot 1$

f real $\to |F|$ even, $\angle F$ odd
f real, even $\to F$ real, even; $F_{-n} = F_n = F_n^*$
f real, odd $\to F$ imaginary, odd; $-F_{-n} = F_n = -F_n^*$

$f_e(t) \to \Re\{F_n\}$
$f_o(t) \to j\,\Im\{F_n\}$

Common FS

$(A=1,\ T=2\pi,\ \omega=1)$
Square $\frac{4}{\pi}(\cos t - \frac{1}{3}\cos 3t + \frac{1}{5}\cos 5t - \ldots)$
$\frac{4}{\pi}(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \ldots)$

Triangle $\frac{8}{\pi^2}(\sin t - \frac{1}{9}\sin 3t + \frac{1}{25}\sin 5t - \ldots)$
$\frac{8}{\pi^2}(\cos t + \frac{1}{9}\cos 3t + \frac{1}{25}\cos 5t + \ldots)$

Sawtooth $\frac{2}{\pi}(\sin t - \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t - \ldots)$
$\frac{2}{\pi}(-\sin t - \frac{1}{2}\sin 2t - \frac{1}{3}\sin 3t - \ldots)$

δ train $\delta_{T_0}(t) = \sum_{n=-\infty}^{\infty}\delta(t-nT_0)$
$\frac{1}{T_0}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}$

FT

Let $F(\omega) \equiv \int f(t)e^{-j\omega t}\,dt$

$F_n = \frac{1}{T_0}\int_{T_0} f(t)e^{-jn\omega_0 t}\,dt$

Limit as $\omega_0 = \Delta\omega \to 0$,
$F_n = \frac{\Delta\omega}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-jn\Delta\omega t}\,dt \equiv \frac{\Delta\omega}{2\pi}F(n\Delta\omega)$

$f_{T_0}(t) = \sum F_n\, e^{jn\omega_0 t} = \sum \frac{\Delta\omega}{2\pi}F(n\Delta\omega)\,e^{jn\Delta\omega t}$

$f(t) = \lim_{T_0\to\infty} f_{T_0}(t) = \frac{1}{2\pi}\int F(\omega)e^{jt\omega}d\omega$

$F(\omega) = |F(\omega)|\,e^{j\angle F(\omega)}$

Real signals: amp and phase symmetry

Existence: weak: energy signal ($|e^{-j\omega t}| = 1$)
Strong: fin num max/min/discont

FT Table

$\delta(t) \to 1$
$1 \to 2\pi\delta(\omega)$

$e^{j\omega_0 t} \to 2\pi\delta(\omega-\omega_0)$
$\cos\omega_0 t \to \pi[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]$
$\sin\omega_0 t \to j\pi[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)]$

$\sum\delta(t-nT_0) \to \omega_0\sum\delta(\omega-n\omega_0)$ $\quad \omega_0 = \frac{2\pi}{T_0}$

$e^{-at}\,u(t) \to \frac{1}{a+j\omega}$ $\quad a>0$
$e^{-a|t|} \to \frac{2a}{a^2+\omega^2}$ $\quad a>0$
$u(t) = \lim_{a\to 0} e^{-at}u(t) \to \lim\frac{1}{a+j\omega}$
$= \lim(\frac{a}{a^2+\omega^2} - j\frac{\omega}{a^2+\omega^2}) = \pi\delta(\omega) + \frac{1}{j\omega}$
$\text{sgn}(t) \to \frac{2}{j\omega}$

$t^n e^{-at}u(t) \to \frac{n!}{(a+j\omega)^{n+1}}$ $\quad a>0$

$\cos\omega_0 t\,u(t) \to \frac{\pi}{2}(\delta(\omega-\omega_0)+\delta(\omega+\omega_0)) + \frac{j\omega}{\omega_0^2-\omega^2}$
$\sin\omega_0 t\,u(t) \to \frac{\pi}{2j}(\delta(\omega-\omega_0)-\delta(\omega+\omega_0)) + \frac{\omega_0}{\omega_0^2-\omega^2}$
$e^{-at}\cos\omega_0 t\,u(t) \to \frac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$ $\quad a>0$
$e^{-at}\sin\omega_0 t\,u(t) \to \frac{\omega_0}{(a+j\omega)^2+\omega_0^2}$ $\quad a>0$

$\text{rect}(\frac{t}{\tau}) \to \tau\,\text{sinc}(\frac{\tau}{2}\omega)$
$\frac{W}{\pi}\,\text{sinc}(Wt) \to \text{rect}(\frac{\omega}{2W})$

$\triangle(\frac{t}{\tau}) \to \frac{\tau}{2}\,\text{sinc}^2(\frac{\tau}{4}\omega)$
$\frac{W}{2\pi}\,\text{sinc}^2(\frac{W}{2}t) \to \triangle(\frac{\omega}{2W})$

$[\omega^2 \text{r}\,(\frac{\omega}{2\omega_0})] \leftarrow \frac{1}{2\pi}\frac{e^{j\omega t}}{(jt)^3}(-\omega^2t^2 - 2j\omega t + 2)_{-\omega_0}^{\omega_0}$
$= \frac{(\omega_0^2t^2-2)\sin\omega_0 t + 2\omega_0 t\cos\omega_0 t}{\pi t^3}$

$[\frac{|\omega|}{\omega_0}\,\text{rect}(\frac{\omega}{2\omega_0})] \leftarrow \frac{\cos\omega_0 t + \omega_0 t\sin\omega_0 t - 1}{\omega_0\pi t^2}$

Frequency domain prop

Linearity

Time shift $f(t-t_0) \to F(\omega)e^{-jt_0\omega}$
$|F|$ unchanged; $\angle F = -t_0\omega$, lin shift

Freq shift $f(t)e^{j\omega_0 t} \to F(\omega-\omega_0)$

Duality $f(t) \to F(\omega)$, $F(t) \to 2\pi f(-\omega)$
pf. $f(t) = \frac{1}{2\pi}\int F(\lambda)e^{jt\lambda}\,d\lambda$
$2\pi f(-t) = \int F(\lambda)e^{-tj\lambda}\,d\lambda = \mathcal{F}[F(\lambda)]$

Reversal $f(-t) \to F(-\omega)$
Scaling $f(at) \to \frac{1}{|a|}F(\frac{\omega}{a})$

Convolution $f*g \to FG$, $fg \to \frac{1}{2\pi}F*G$
$\mathcal{F}[f*g] = \int e^{-j\omega t}\int f(\tau)g(t-\tau)d\tau dt$
$= \int f(\tau)\mathcal{F}[g(t-\tau)]d\tau$
$= \int f(\tau)G(\omega)e^{-j\omega\tau}d\tau$
$\frac{1}{2\pi}\mathcal{F}^{-1}[F*G] = (\frac{1}{2\pi})^2\int e^{j\omega t}\int F(\lambda)G(\omega-\lambda)d\lambda d\omega$

Diff $f^{(n)}(t) \to (j\omega)^n F(\omega)$ (diff $e^{j\omega t}$)

Int $\int_{-\infty}^{t} f(\tau)d\tau \to \frac{1}{j\omega}F(\omega) + \pi F(0)\delta(\omega)$
$\int = f(t)*u(t) \to F(\omega)U(\omega)$
$U(\omega) = \lim\frac{1}{a+j\omega} = \lim(\frac{a}{a^2+\omega^2} - j\frac{\omega}{a^2+\omega^2})$
$= \pi\delta(\omega) + \frac{1}{j\omega}$ $(\int\frac{a}{\omega^2+a^2}d\omega = \tan^{-1} = \pi)$

Conjugation $f(t)^* \to F(-\omega)^*$

Symmetry Re $\to$ mag even, phase odd ($F(-\omega) = F(\omega)^*$)
real, even $\to$ real, even; real, odd $\to$ imaginary, odd

f even: $F(\omega) = 2\int_0^{\infty} f(t)\cos(\omega t)dt$
f odd: $F(\omega) = -2j\int_0^{\infty} f(t)\sin(\omega t)dt$

Parseval

$E_f = \int |f(t)|^2 dt = \frac{1}{2\pi}\int |F(\omega)|^2 d\omega$ for energy sig

$= \int f f^* dt = \int f(t)\mathcal{F}^{-1}[F(-\omega)^*]dt$

$= \int f(t)\frac{1}{2\pi}\int F(-\omega)^* e^{j\omega t} d\omega dt$

$= \frac{1}{2\pi}\int f(t)\int F(\lambda)^* e^{-jt\lambda} d\lambda dt$

$= \frac{1}{2\pi}\int F(\lambda)^* \int f(t) e^{-j\lambda t} dt d\lambda$

$\Delta E_f = \frac{2}{2\pi}\int_{\omega_1}^{\omega_2} |F(\omega)|^2 d\omega$

Autocorrelation $\psi_f(t) \equiv \int f(\tau)f(\tau - t)d\tau \to |F(\omega)|^2$

Modulation

$m(t)\cos(\omega_c t) \to \frac{1}{2}[M(\omega+\omega_c) + M(\omega-\omega_c)]$

$e(t) = m(t)\cos^2\omega_c t$

$E(\omega) = \frac{1}{2}M + \frac{1}{4}[M(\omega+2\omega_c) + M(\omega-2\omega_c)] \to$ LPF

SSB 1/4 gain

$\phi_{\text{AM}}(t) = [A + f(t)]\cos(\omega_0 t)$

$A \ge f(t)$ for all t

modulation index $\mu \equiv f_{\max}/A$

$\mu = \infty$, suppressed carrier; $\mu = 1$, marginal

LTIC sys trans, (marginally) stable

Let $e^{j\omega t} \Rightarrow H(\omega)e^{j\omega t}$

$\lim\sum \frac{F(n\Delta\omega)\Delta\omega}{2\pi} e^{jn\Delta\omega t} \Rightarrow \lim\sum \frac{F(n\Delta\omega)H(n\Delta\omega)\Delta\omega}{2\pi} e^{jn\Delta\omega t}$

$y(t) = \frac{1}{2\pi}\int F(\omega)H(\omega)e^{j\omega t}d\omega$

$Y(\omega) = F(\omega)H(\omega)$

Distortionless $y(t) = kf(t-t_d)$, so $H(\omega) = ke^{-j\omega t_d}$

Payley-Wiener H realizable, h causal iff

$\int \frac{|\ln|H(\omega)||}{1+\omega^2}d\omega < \infty$ (consecutive 0s)

Truncate $\hat{h}(t) = h(t)u(t)$

Periodic FT

$f(t) = \sum_n F_n e^{jn\omega_0 t}$

$\mathcal{F}[f(t)] = 2\pi\sum_n F_n \delta(\omega - n\omega_0)$

$Y = F(\omega)H(\omega) = 2\pi\sum F_n H(n\omega_0)\delta(\omega - n\omega_0)$

$Y_n \equiv F_n H(n\omega_0)$. Periodic with same ω_0

Eigen: $f(t) = e^{j\omega_0 t}$, $Y_1 = H(1\omega_0)$, $y(t) = H(1\omega_0)e^{j\omega_0 t}$

$f(t) = \cos(\omega_0 t + \theta)$, assume $h(t)$ real

$y = \frac{1}{2}(e^{j(\theta+\omega_0 t)}H(\omega_0) + e^{-j(\theta+\omega_0 t)}H(-\omega_0))$

$= |H(\omega_0)|\cos(\omega_0 t + \theta + \angle H(\omega_0))$

$\cos 2t * e^{-3t}u(t) \equiv f * h$

$= |H(2)|\cos(2t + \angle H(2))$

Sampling

$\overline{f}(t) \equiv f(t)\delta_{T_s}(t) = \sum f(nT_s)\delta(t - nT_s)$

$\overline{F}(\omega) = \frac{1}{2\pi}F(\omega) * [\frac{2\pi}{T_s}\sum\delta(\omega - n\omega_s)] = \frac{1}{T_s}\sum F(\omega - n\omega_s)$

$\omega_s \ge 4\pi B$, $F_s \ge F_N \equiv 2B$

Intrapolation when $F_s = 2B$

$F(\omega) = \overline{F}(\omega)T_s\,\text{rect}(\frac{\omega}{4\pi B})$

If $F_s = 2B$, $f(t) = \overline{f}(t) * \frac{2B}{F_s}\text{sinc}(2\pi Bt)$

$= \sum_n f(nT_s)\delta(t - nT_s) * \text{sinc}(2\pi Bt)$

$= \sum_n f(nT_s)\,\text{sinc}(2\pi Bt - n\pi)$

ana FS, with basis sinc, weighted sample sum

If $F_s > 2B$, $f(t) = \sum f(nT_s)w(t - nT_s)$ not sinc weight

for some relaxed LP filter $w(t)$

Anti-alias before sampling: LPF of $F_s/2$

Practical sampling

$p_T(t) = \frac{\tau}{T_s} + \sum(\frac{2}{\pi n}\sin(n\pi\frac{\tau}{T_s}))\cos(n\omega_s t)$

$P_T(\omega) = 2\pi\frac{\tau}{T_s}\delta(\omega) + \sum\frac{\pi 2\sin(\ldots)}{\pi n}[\delta(\omega + n\omega_s) + \delta(\omega - n\omega_s)]$

LT

$\mathcal{L}[-e^{-at}u(-t)] = \mathcal{L}[e^{-at}u(t)]$, except ROC

If sig are causal, $\mathcal{L}$ is 1-to-1

Unilateral: $\mathcal{L}[f] = \int_{0^-}^{\infty} f(t)e^{-st}dt$

$= \int[f(t)e^{-\sigma t}]e^{-j\omega t}dt$

σ_0: smallest σ to make integral converge

Uni LT Table

Watch ROC!!

$\delta(t) \to 1$ — $\forall s$

$u(t) \to \frac{1}{s}$ — $\Re(s) > 0$

$t^n u(t) \to \frac{n!}{s^{n+1}}$

$e^{\lambda t}u(t) \to \frac{1}{s-\lambda}$ — $\Re(s) > \Re(\lambda)$

$t^n e^{\lambda t}u(t) \to \frac{n!}{(s-\lambda)^{n+1}}$

$\frac{1}{(n-1)!}t^{n-1}e^{\lambda t}u(t) \to \frac{1}{(s-\lambda)^n}$

$e^{-at}\cos(bt)\,u(t) \to \frac{s+a}{(s+a)^2+b^2}$

$e^{-at}\sin(bt)\,u(t) \to \frac{b}{(s+a)^2+b^2}$

$re^{-at}\cos(bt+\theta)\,u(t) \to \frac{(r\cos\theta)s + (ar\cos\theta - br\sin\theta)}{s^2+2as+(a^2+b^2)}$

$2re^{-at}\cos(bt+\theta)\,u(t) \to \frac{re^{j\theta}}{s-(-a+jb)} + \frac{re^{-j\theta}}{s-(-a-jb)}$

$e^{-at}\left[A\cos(bt) + \frac{B-Aa}{b}\sin(bt)\right]u(t)$

$= \frac{\sqrt{A^2c+B^2-2ABa}}{b}e^{-at}\cos\left(bt + \tan^{-1}\frac{Aa-B}{Ab}\right)u(t)$

$\to \frac{As+B}{s^2+2as+c}$ — $b \equiv \sqrt{c-a^2}$

LT Prop

Linearity — ROC $\cap$

Time delay $f(t-t_0)\,u(t-t_0) \to F(s)e^{-st_0}$ — ROC same

$t_0 > 0$; pf: $\int_{-t_0}^{\infty}$

Freq shift $f(t)e^{s_0 t} \to F(s-s_0)$ — $\Re(s) > \sigma_0 + \Re(s_0)$

Scaling $(a > 0)$, $f(at) \to \frac{1}{|a|}F(\frac{s}{a})$ — $\Re(s) > a\sigma_0$

Convolution $f_1 * f_2 \to F_1F_2$ — ROC $\cap$

$f_1f_2 \to \frac{1}{2\pi j}F_1 * F_2$

Time diff $\dot{f}(t) \to sF(s) - f(0^-)$ — $\Re(s) > \max(\sigma_0, 0)$

$\ddot{f}(t) \to s^2F(s) - sf(0^-) - \dot{f}(0^-)$

pf (parts): $\int_{0^-}^{\infty}\dot{f}(t)e^{-st}dt = [f(t)e^{-st}]_{0^-}^{\infty} + sF(s)$

Time int $\int_{0^-}^{\infty} f(\tau)d\tau \to \frac{1}{s}F(s)$ pf: diff

Freq diff $-tf(t) \to \frac{dF(s)}{ds}$

Freq int $\frac{1}{t}f(t) \to \int_s^{\infty}F(z)dz$

IVT $f(0^+) = \lim_{s\to\infty} sF(s)$ — if exists

pf: $\mathcal{L}[\dot{f}(t)] = \int_{0^-}^{\infty}\dot{f}(t)e^{-st}dt$

$sF(s) - f(0^-) = \int_{0^-}^{0^+}\dot{f}(t)e^{-st}dt + \int_{0^+}^{\infty}\dot{f}(t)e^{-st}dt$

$sF(s) - f(0^-) = f(0^+) - f(0^-)$

FVT $f(\infty) = \lim_{s\to 0} sF(s)$ — if exists

pf: $\mathcal{L}[\dot{f}(t)] = \int_{0^-}^{\infty}\dot{f}(t)e^{-st}dt$

$\lim_{s\to 0} sF(s) - f(0^-) = \lim_{s\to 0}\int_{0^-}^{\infty}\dot{f}(t)e^{-st}dt$

$sF(s) - f(0^-) = f(\infty) - f(0^-)$

Rational $\mathcal{L}^{-1}$

First rationalize

$F(s) = \frac{b_m s^m + \ldots + b_1 s + b_0}{1s^n + \ldots + a_1 s + a_0} \equiv \frac{P(s)}{Q(s)}$

$= \frac{a_0}{(s-\lambda)^r} + \ldots + \frac{a_{r-1}}{s-\lambda} + \frac{k_1}{s-\lambda_1} + ..$

$k = (s-\lambda_i)F(s)|_{s=\lambda_i}$

$a_0 = (s-\lambda)^r F(s)|_{s=\lambda}$

$a_m = \frac{1}{m!}\frac{d^m}{ds^m}[(s-\lambda)^r F(s)]|_{s=\lambda}$

$\mathcal{L}^{-1}[\frac{1}{(s-\lambda)^n}] = \frac{1}{(n-1)!}t^{n-1}e^{\lambda t}u(t)$

Multiply both by s and let $s = \infty$, only $\frac{1}{s}$ term left

Sys Anal

$Q(D)y(t) = P(D)f(t)$

$H(s) = \frac{Y_{zs}(s)}{F(s)} = \frac{P(s)}{Q(s)}$

Asym. (internal, init): $y_{zi}(t) \to 0$ as $t \to \infty$
marginal: $y_{zi}(t)$ remains bounded (unique λs on Im axis)

BIBO (external, input) iff $\int_{-\infty}^{\infty} |h(t)|\, dt$ exists

$\Rightarrow |f(t)| < K$

$y_{zs}(t) = h * f = \int h(\tau) f(t-\tau)\, d\tau$

$\leq \int |h(\tau)||f(t-\tau)| d\tau < K \int |h(\tau)|\, d\tau$

$\Leftarrow$ Let $f(t) = \operatorname{sgn}(h(-t))$

$y(0) = \int h(\tau) f(0-\tau) d\tau$

$= \int h(\tau) \operatorname{sgn}(h(\tau)) d\tau$

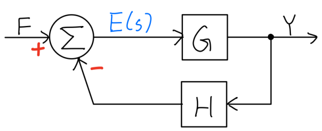
$= \int |h(\tau)| d\tau \equiv \infty$

Asy stable $\Rightarrow$ BIBO stable (all exp)
Marginal $\Rightarrow$ BIBO unstable ($\int |\sin(t)|\, dt = \infty$)

Sys Realization

Feedback: $Y = GE = G(F - HGY)$

$H_{\text{eff}} = \frac{Y}{F} = \frac{G}{1+HG}$



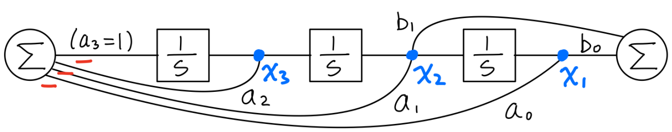
Canonical

$F(s) = 1s^3X + a_2 s^2 X + a_1 sX + a_0 X$

$s^3 X = F - a_2 s^2 X - a_1 sX - a_0 X$

$b_3 s^3 X + b_2 s^2 X + b_1 sX + b_0 X = Y(s)$

$H = \frac{b_1 s + b_0}{1s^3 + a_2 s^2 + a_1 s + a_0}$



$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -a_0 & -a_1 & -a_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix} f$$

$y = \begin{bmatrix}b_0 & b_1 & 0\end{bmatrix} \cdot \mathbf{x}$

Second form:

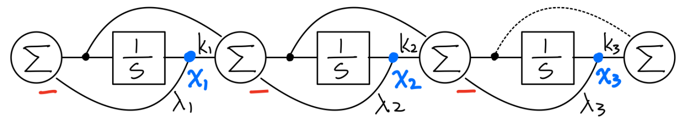
$s^2 Y = s^2 b_2 F + s(-a_1 Y + b_1 F) + (-a_o Y + b_0 F)$

$Y = b_2 F + \frac{1}{s}(-a_1 Y + b_1 F) + \frac{1}{s^2}(-a_o Y + b_0 F)$

Transpose $\mathbf{A}$, swap and trans $\mathbf{b}$, $\mathbf{c}$

Cascade (made $P_3(s)$ constant here)

$H = \frac{P_1(s)}{s+\lambda_1} \cdot \frac{P_2(s)}{s+\lambda_2} \cdot \frac{k_3}{s+\lambda_3}$

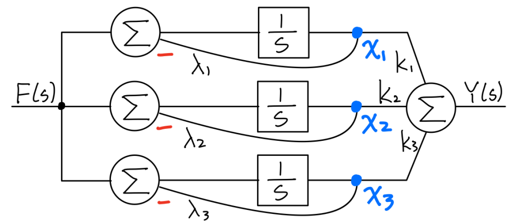


$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-\lambda_1 & 0 & 0\\ .. & -\lambda_2 & 0\\ .. & .. & -\lambda_3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}1\\ ..\\ ..\end{bmatrix} f$$

$y = \begin{bmatrix}0 & 0 & k_3\end{bmatrix} \cdot \mathbf{x}$

Parallel

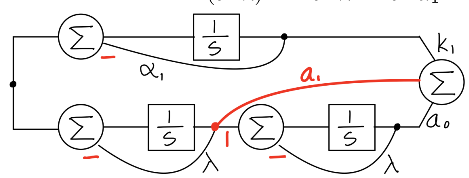
$H = \frac{k_1}{s+\lambda_1} + \frac{k_2}{s+\lambda_2} + \frac{k_3}{s+\lambda_3}$



$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-\lambda_1 & 0 & 0\\ 0 & -\lambda_2 & 0\\ 0 & 0 & -\lambda_3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}1\\ 1\\ 1\end{bmatrix} f$$

$y = \begin{bmatrix}k_1 & k_2 & k_3\end{bmatrix} \cdot \mathbf{x}$

(same pole) $H = \frac{a_0}{(s-\lambda)^2} + \frac{a_1}{s-\lambda} + \frac{k_1}{s-\alpha_1}$



State Equations

Def: *state* of a sys at any time t_0 is the *smallest* set of nums $\{x_i(t_0)\}$ that is sufficient to detemrine sys behavior $\forall t > t_0$ when input $f(t)$ is known for $t > t_0$

Always $\boxed{\int}$ output

If $t_0 = 0$, initial condition

$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}f$

$y = \mathbf{c} \cdot \mathbf{x}$

MIMO

$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{f}$

$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{f}$

$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{F}(s)$

$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{x}(0) + \mathbf{B}\mathbf{F}(s)$

$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{F}(s)$

$\mathbf{x}(t) = \mathbf{x}_{zi}(t) + \mathbf{x}_{zs}(t)$

LTI Freq Response

$H(\omega) = H(s)|_{s=j\omega} = \frac{P(s)}{Q(s)}$

Ideal delay: $|H| = 1, \angle H(\omega) = -\omega T$
Ideal diff: $|H(\omega)| = |\omega|, \angle H(\omega) = \pm\frac{\pi}{2}$
Ideal int: $|H(\omega)| = \frac{1}{|\omega|}, \angle H(\omega) = \mp\frac{\pi}{2}$

$f(t) = C = Ce^{0t}$

$y(t) = H(0)C$

$f(t) = e^{st}$ everlasting

$y(t) = h(t) * e^{st} = e^{st} \int h(\tau) e^{-s\tau} d\tau \equiv H(s)e^{st}$

$f(t) = e^{j\omega_0 t}\, u(t)$

$Y_{zs}(s) = F(s)H(s)$

$= \frac{1}{s - j\omega_0} \frac{P(s)}{(s-\lambda_1)\ldots(s-\lambda_n)}$

$= \frac{H(s)|_{s=j\omega_0}}{s-j\omega_0} + \frac{k_1}{s-\lambda_1} + \ldots$

$y_{zs}(t) = H(\omega_0)e^{j\omega_0 t}\, u(t) + \sum_i k_i e^{\lambda_i t}\, u(t)$

$y_{ss}(t)$ scaled input
$y_{tr}(t)$ decays for stable sys

$f(t) = \cos(\omega_0 t + \theta)\, u(t)$

$y_{ss}(t) = |H(\omega_0)| \cos(\omega_0 t + \theta + \angle H(\omega_0))\, u(t)$

Pole Zero

$H(j\omega) = b_n \frac{(j\omega - z_1)..(j\omega - z_n)}{(j\omega - p_1)..(j\omega - p_n)}$

$\equiv b_n \frac{(r_1 e^{j\phi_1})\ldots(r_n e^{j\phi_n})}{(d_1 e^{j\theta_1})\ldots(d_n e^{j\theta_n})}$

$|H(\omega)| = b_n \frac{r_1 \ldots r_n}{d_1 \ldots d_n}$

$\angle H(\omega) = (\phi_1 + .. + \phi_n) - (\theta_1 + .. + \theta_n)$

Vector from p/z to $j\omega$
Pole enhances gain; Zero suppresses