Fourier

Vector $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $\underline{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

inner product $\langle \underline{x}, \underline{y} \rangle = \underline{x}^T \underline{y} = \sum_{i=1}^{n} x_i y_i$ (Orthogonal if $=0$)

norm $\|\underline{x}\| = \sqrt{\langle \underline{x}, \underline{x} \rangle}$ $\frac{\underline{x}}{\|\underline{x}\|} \rightarrow$ unit norm (norm $=1$)

ortho. + both unit norm → orthonormal

Orthonormal basis $\{\underline{v_1}, \cdots \underline{v_n}\}$ s.t. $\langle \underline{v_i}, \underline{v_j} \rangle \neq 0 \quad i \neq j$

$\langle \underline{v_i}, \underline{v_j} \rangle = 1 \quad i = j$

Then for any vector $\underline{x} \in \mathbb{R}^n$, $\underline{x} = \sum c_i \underline{v_i}$,

where $c_i = \langle \underline{x}, \underline{v_i} \rangle$ Projection of $\underline{x}$ on $\underline{v_i}$

Signal rep. by ornor signal set

1. Real sig. Let $\{x_1(t), \cdots x_N(t);\ t \in [t_1, t_2]\}$ be a set of ortho sig.

s.t. $\langle x_i, x_j \rangle = 0, \quad i = j$ where $\langle x_i, x_j \rangle = \int_{t_1}^{t_2} x_i(t) x_j(t) dt$

$= \overset{\text{energy}}{\mathcal{E}_i} \quad i = j \rightarrow \int (x_i^2) dt$

if $\mathcal{E}_i = 1 \ \forall i$, then set is orthonormal.

Approximate any sig. $f(t)$, $t \in [t_1, t_2]$ by ortho. set.

$$\hat{f}(t) = \sum_{n=1}^{N} c_n x_n(t), \quad t \in [t_1, t_2]$$

Approach: minimize error $e(t) \equiv f(t) - \sum_{n=1}^{N} c_n x_n(t)$

(ana projection

$$\mathcal{E}_e \equiv \int_{t_1}^{t_2} [e(t)]^2 dt$$

$$= \int_{t_1}^{t_2} \left(f(t) - \sum_{i=1}^{N} c_i x_i(t) \right)^2 dt$$

$$= \int_{t_1}^{t_2} (f(t))^2 dt - 2 \sum_{i=1}^{N} c_i \int_{t_1}^{t_2} f(t) x_i(t)\, dt + \int_{t_1}^{t_2} \left(\sum_{i=1}^{N} c_i x_i(t) \right)^2 dt$$

$$= \mathcal{E}_f - 2 \sum_{i=1}^{N} c_i \langle f(t), x_i(t) \rangle + \sum_{i=1}^{N} c_i^2 \int_{t_1}^{t_2} x_i(t)^2 dt + \sum_{i \neq j} c_i c_j \cancel{\int_{t_1}^{t_2} x_i(t) x_j(t) dt}$$

$$= \mathcal{E}_f - 2 \sum_{i=1}^{N} c_i \langle f(t), x_i(t) \rangle + \sum_{i=1}^{N} c_i^{\textcircled{2}} \mathcal{E}_i \rightarrow \text{Quadratic}$$

Min $\frac{\partial \mathcal{E}_e}{\partial c_i} = 2 \mathcal{E}_i c_i - 2 \langle f(t), x_i(t) \rangle + 0 = 0$

$$c_i = \frac{1}{\mathcal{E}_i} \langle f(t), x_i(t) \rangle \quad (f \text{ proj. on } x_i)$$

$$\boxed{\mathcal{E}_e^{\min} = \mathcal{E}_f - \sum_{i=1}^{N} \mathcal{E}_i c_i^2}$$

If $\mathcal{E}_e^{\min} \rightarrow 0$ as $n \rightarrow \infty$, $\overset{\{x_i(t)\}}{\text{basis}}$ is complete. $\{x_i(t)\}$ is basis signals

$$f(t) = \hat{f}(t) = \sum_{i=1}^{\infty} c_i x_i(t), \quad t \in [t_1, t_2]$$

1. Trignometric set

$$\{\overset{DC}{1},\ \cos \overset{\text{fund. freq.}}{\omega_0} t,\ \cos 2\omega_0 t,\ \cdots,\ \cos n\omega_0 t,$$
$$\sin \omega_0 t,\ \cdots\ \sin n\omega_0 t\}$$

Orthogonality (over duration $\frac{2\pi}{\omega_0}$)

Pf. - $\langle 1, \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} n\omega_0 t\rangle$, $\int = 0$

- $\langle \cos n\omega_0 t, \overset{n\neq m}{\cos} m\omega_0 t\rangle$, $\int_{T_0} \cos n\omega_0 t \cos m\omega_0 t\, dt$

$$= \int_{T_0} \frac{1}{2}\left[\cos (n+m)\omega_0 t + \cos (n-m)\omega_0 t\right] dt = 0$$

- sin n sin m same
- $\langle \sin n\omega_0 t, \cos m\omega_0 t\rangle \overset{\cdots}{\rightarrow}$ same

Energy - 1 $\rightarrow \int_{T_0} 1^2 dt = T_0$

- else $\int_{T_0} \cos^2 n\omega_0 t\, dt \overset{\cdots}{=} \frac{T_0}{2}$

Completeness Pf. omitted

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \qquad t_0 \le t \le t_0 + T_0 \equiv \frac{2\pi}{\omega_0}$$

$$a_0 = \frac{1}{T_0}\langle f(t), 1\rangle = \frac{1}{T_0}\int_{T_0} f(t)\,dt$$
$$a_n = \text{——} = \frac{2}{T_0}\int_{T_0} f(t) \cos n\omega_0 t\, dt$$
$$b_n = \text{——} = \frac{2}{T_0}\int_{T_0} f(t) \sin n\omega_0 t\, dt$$

If f(t) periodic, no restriction on t.

Symmetry : $\int_{-T_0/2}^{T_0/2}$, if f is

	even	odd
$a_0 =$	$\frac{2}{T_0}\int_0^{T_0/2} f(t)\, dt$	0
$a_n =$	$\frac{4}{T_0}\int_0^{T_0/2} f(t) \cos n\omega_0 t\, dt$	0
$b_n =$	0 (——)	$\frac{4}{T}\int_0^{T_0/2}$ ——

Compact trig Fourier $a_n \cos n\omega_0 t + b_n \sin n\omega_0 t = C_n \cos(n\omega_0 t + \theta_n)$

$$C_n = \sqrt{a_n^2 + b_n^2}, \quad \theta_n = \pm \tan^{-1}\left(\frac{-b_n}{a_n}\right)$$
$$C_0 = a_0$$

$$f(t) = C_0 + \sum_{t=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n) \qquad C_n \geq 0$$

Freq. domain C_n vs n magnitude spectrum

θ_n vs n phase ——

Existence condition for Fourier

1. Weak if periodic sig. satisfies $\int_{T_0} f(t)^2 dt < \infty$, (finite power)

then the F. coeff. a_n, b_n are fin., $\mathcal{E}_e \to 0$ as $n \to \infty$

Not specifying if $f(t) \overset{?}{=}$ series $\forall t$

2. Strong a) $\sim\sim\sim$ $\int_{T_0} |f(t)| dt < \infty$

b) $f(t)$ has a fin # of min/max in T_0 } reasonably smooth

c) $\sim\sim\sim$ discontinuities in T_0 }

then $f(t)$ = series $\forall t$ s.t. $f(t)$ is cont.

$\frac{f(t_0^+) + f(t_0^-)}{2}$ = series $\forall$ discont. t_0 E. 

periodic + periodic Let $f_1(t) = f_1(t+T_1)$, $f_2(t) = f_2(t+T_1)$

Let sum $g(t) = f_1(t) + f_2(t)$ is periodic, $g(t) = g(t+T_0)$

$$f_1(t) + f_2(t) = f_1(t+T_0) + f_2(t+T_0)$$

$$= f_1(t+T_1) + f_2(t+T_2) \longrightarrow T_0 = mT_1 + nT_2$$

$T_0 = LCM(T_1, T_2)$ $\omega_0 = GCD(\omega_1, \omega_2)$ $\frac{T_1}{T_2} = \frac{n}{m} \to$ rational

When two ω ratios are rational $\to$ harmonically related

E. $f(t) = 2 + \cos(\frac{1}{2}t + \theta_1) + 3\cos(\frac{2}{3}t + \theta_2) + 5\cos(\frac{7}{6}t + \theta_3)$

$T_1, T_2, T_3 = 4\pi, 3\pi, \frac{12\pi}{7} \to T_0 = 12\pi$

Cmplx-valued sig. Need def. inner product $\langle x(t), y(t) \rangle = \int x(t)\, y(t)^{\overset{\text{conj.}}{*}} dt$

$\{x_n\}$ is orthogonal if $\langle x_n(t), x_m(t) \rangle$ = - - - - 0, $\mathcal{E}_n$ (same)

if $\{x_n\}$ is complete, $f(t) = \sum_n c_n x_n(t)$, where $c_n = \frac{1}{\mathcal{E}_n} \langle f(t), x_n(t) \rangle$

$$= \frac{1}{\mathcal{E}_n} \int f(t)\, x_n(t)^* dt$$

(bounds are fin. interval $[t_1, t_2]$)

2. Complex exp. set $\{e^{jn\omega_0 t}; n = 0, \pm 1, \pm 2, \dots\}$ 2-sided

Pf. $n=m$ $\int_T e^{jn\omega_0 t} e^{-jn\omega_0 t} dt = \int_T 1\, dt = T \equiv \mathcal{E}_n$

$n \neq m$ $\int_T e^{jn\omega_0 t} e^{-jm\omega_0 t} dt = \int_T e^{j(n-m)\omega_0 t} dt$

$$= \frac{1}{j(n-m)\omega_0} \cancel{[e^{\sim\sim} - e^{\sim\sim}]} = 0$$

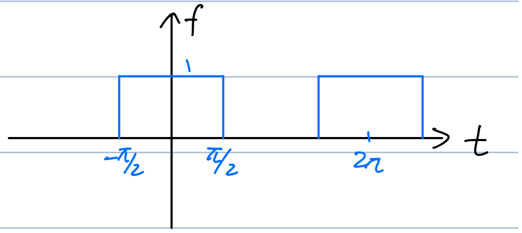
periodic!

Exp Fourier $f(t)=\sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$, where $F_n = \frac{1}{T_0}\langle f(t), e^{jn\omega_0 t}\rangle$

$= \frac{1}{T_0}\int_{T_0} f(t)\, e^{-jn\omega_0 t}\, dt$

No sym. to exploit :(

E. $T_0 = 2\pi$, $\omega_0 = 1$



$F_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f(t)\, e^{-jn\omega_0 t}\, dt$

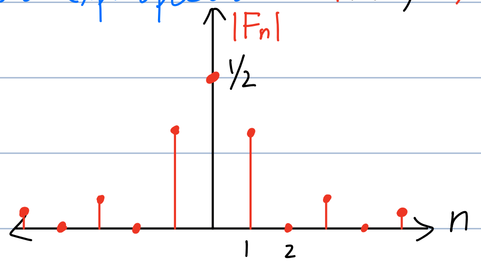
$n \neq 0$ $= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{-jnt}\, dt$

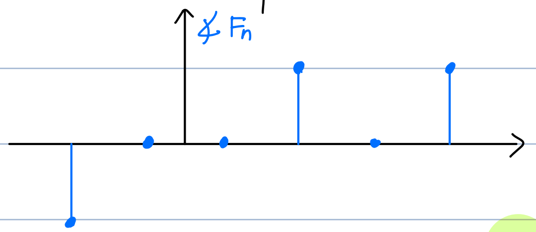
$= -\frac{1}{2\pi jn}\left(e^{-\frac{jn\pi}{2}} - e^{\frac{jn\pi}{2}}\right)$

$= \frac{1}{\pi n}\,\frac{e^{\frac{jn\pi}{2}} - e^{-\frac{jn\pi}{2}}}{2j}$ $= \frac{1}{\pi n}\sin\left(\frac{\pi}{2}n\right)$

$n = 0$, $F_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, dt = \frac{1}{2}$

Plot exp. spectra : $|F_n|$, $\measuredangle F_n$, $n \in \mathbb{Z}$ special case: all real!





even!

odd! $(-\pi = \pi)$

vs. trig half & mirrored negated & mirrored.

$C_n \to F_n$ $C_n\cos(n\omega_0 t + \theta_n) = \frac{C_n}{2}\left(e^{j(n\omega_0 t + \theta_n)} + e^{-j(n\omega_0 t + \theta_n)}\right)$

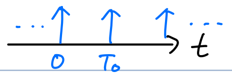
$= \underbrace{\left(\frac{C_n}{2}e^{j\theta_n}\right)}_{F_n} e^{jn\omega_0 t} + \underbrace{\left(\frac{C_n}{2}e^{-j\theta_n}\right)}_{F_{-n}} e^{-jn\omega_0 t}$

even odd

$(F_0 = C_0 = A_0)$ $F_{-n} \to |F_n| = |F_{-n}| = \frac{C_n}{2}$, $\measuredangle F_n = \theta_n$ $\measuredangle F_{-n} = -\theta_n$

$F_n \to C_n$ $C_0 = F_0$, $C_n = 2|F_n|$, $\theta_n = \measuredangle F_n$

E. $f(t) = \sum_{n=-\infty}^{\infty}\delta(t - nT_0) \equiv \delta_{T_0}(t)$



pulse train

$\omega_0 = \frac{2\pi}{T_0}$ $F_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2}\delta_{T_0}(t)\, e^{-jn\omega_0 t}\, dt$

$= \frac{1}{T_0}\cancel{e^{0}}$

$f(t) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}$ $|F_n| = \frac{1}{T_0}$, $\measuredangle F_0 = 0$, $C_n = a_n = \frac{2}{T_0}$, $\theta_n = 0$, $b_n = 0$

Freq. domain prop. $\overset{\text{time dom.}}{f(t)} \leftrightarrow \overset{\text{freq. dom.}}{\{F_n\}}$

① Linearity if (same period) $f(t) \leftrightarrow F_n,\ g(t) \leftrightarrow G_n \Rightarrow a f(t) + b g(t) \leftrightarrow aF_n + bG_n$

② Time shift if $f(t) \leftrightarrow F_n$, $f(t-t_0) \leftrightarrow \overset{\text{same } |F_n|}{F_n}\, e^{-jn(w_0 t_0)} \rightarrow \measuredangle F_n$ shifted by $\overset{\text{linear phase shift}}{-(w_0 t_0)\textcircled{n}}$

Pf. $G_n = \frac{1}{T_0}\int_{t_1}^{t_1+T_0} f(t-t_0)\, e^{-jnw_0 t}$, $\tau \equiv t_1 - t_0$, — —

Let $\tilde{f}(t) = f(t-t_0) \rightarrow \tilde{F}_n = F_n e^{-j(w_0 t_0)n}$

$\tilde{C}_n = 2|\tilde{F}_n| = 2|F_n| = C_n \quad \tilde{\theta}_n = \theta_n - (w_0 t_0)\, n$

$\tilde{a}_n = \tilde{C}_n \cos\tilde{\theta}_n = a_n \cos(w_0 t_0 n) - b_n \sin(w_0 t_0 n)$

$\tilde{b}_n = -\tilde{C}_n \sin\tilde{\theta}_n = -b_n \cos(w_0 t_0 n) + a_n \sin(w_0 t_0 n)$

③. Time reversal ——, $f(-t) \leftrightarrow \overset{\text{frequency reversal}}{F_{-n}}$ Pf. $\tau \equiv -t$, — ——

E. if $f(t)$ even, $f(-t) = f(t)$, $F_{-n} = F_n \rightarrow$ even in $F \rightarrow \theta_n = 0 \rightarrow$ cos only

- - - $\tilde{C}_n = C_n$, $\tilde{\theta}_n = -\theta_n$; $\tilde{a}_n = a_n$, $\tilde{b}_n = -b_n$

④. Time scaling (T_0 changes!) $T = \frac{T_0}{a}$ $w = a w_0 \rightarrow$ fund. freq. changes. (basis signal)

Pf. $f(at) = \sum_{n=-\infty}^{\infty} F_n e^{jn(w_0 a)t}$

E. $f(t) \leftrightarrow F_n$, $T_0 = 4 \rightarrow f(-t+1) \leftrightarrow ?$

$w = \frac{\pi}{2}$. Let $g(t) = f(t+1)$, $G_n = F_n e^{j\frac{\pi}{2}n}$

$h(t) = g(-t)$, $H_n = G_{-n} = F_{-n}\, e^{-j\frac{\pi}{2}n}$

⑤ Multiplication (same period T_0) $h(t) = f(t)\, g(t)$

Pf. $H_n = \frac{1}{T_0}\int_{T_0} f(t) g(t) e^{-jnw_0 t}\, dt$

$= \frac{1}{T_0}\int_{T_0} \left(\sum_m F_m e^{jmw_0 t}\right)\left(\sum_k G_k e^{jkw_0 t}\right) e^{-jnw_0 t}\, dt$

$= \sum_m \sum_k F_m G_k \cdot \frac{1}{T_0}\int_{T_0} e^{j(m+k-n)w_0 t}\, dt$

$= \sum_m \sum_k F_m G_k \langle e^{j(m+k)w_0 t}, e^{jnw_0 t} \rangle \rightarrow =1$ only if $n = m+k$

$= \sum_k G_k F_{n-k} \equiv \boxed{F_n * G_n}$

⑥ Conjugation $f(t)^* \leftrightarrow F^*_{-n}$ $a^* b^* = (ab)^*$

Pf. $G_n = \frac{1}{T_0}\int_{T_0} f(t)^* e^{-jnw_0 t}\, dt = \left(\frac{1}{T_0}\int_{T_0} f(t)\, e^{jnw_0 t}\, dt\right)^*$

E. If $f(t)$ real, $f(t) = f(t)^* \leftrightarrow F_n = F^*_{-n} \rightarrow F_{-n} = F^*_n \rightarrow |\ |$ even, $\measuredangle$ odd

⑦ Parseval's relation $f(t)=\sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$

n th harmonic $F_n e^{jn\omega_0 t}$. Power $\frac{1}{T_0}\int_{T_0} |f(t)|^2 dt = \sum_{-\infty}^{\infty} |F_n|^2$ conserved

Pf. $= \frac{1}{T_0}\int_{T_0} f(t)\cdot f(t)^* dt$

$= \frac{1}{T_0}\int_{T_0} \left(\sum_n F_n e^{jn\omega_0 t}\right)\left(\sum_m F_m e^{jm\omega_0 t}\right)^* dt$

$= \sum_n \sum_m F_n F_m^* \frac{1}{T_0}\int_{T_0} e^{j(n-m)\omega_0 t} dt \overset{\cdots}{=}$

(Conj. prop.)

⑧ Symmetry Assume f(t) always real

a. even $b_n = 0$ F_n is real & even

$F_{-n} \overset{\text{even}}{=} F_n \overset{\text{real}}{=} F_n^*$

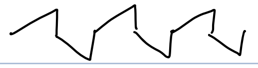
odd $a_n = 0$ F_n is im & odd

Pf. $F_{-n} = F_n^*$, do time reversal

b. $f_e(t) \leftrightarrow \mathrm{Re}\{F_n\}$, $f_o(t) \leftrightarrow j\,\mathrm{Im}\{F_n\}$

Pf. $F_{-n} = F_n^*$,

c. Half-wave symmetry : $f(t-\frac{T_0}{2}) = -f(t)$

E. 

$a_{2k} = b_{2k} = 0$

Pf. HW

E. Find f(t) w/ $\begin{cases} f(t) \text{ is real, periodic w/ } T_0 = 4 \\ |F_n| = 0 \text{ for } |n| > 1 \\ g(t) \text{ w/ } G_n = e^{j\frac{\pi}{2}n} F_{-n} \text{ is odd} \\ P_f = \frac{1}{2} \end{cases}$

$\omega_0 = \frac{\pi}{2}$, $f(t) = F_{-1} e^{-j\frac{\pi}{2}t} + F_0 + F_1 e^{j\frac{\pi}{2}t}$ $(F_{-1} = F_1^*)$

$G_n = e^{-j\frac{\pi}{2}n} F_{-n}$

$g(t) = f(-t+1)$ is real and odd → G_n is im. and odd → $G_0 = 0$, $G_{-1} = -G_1$

$\frac{1}{2} = P_f = P_g = \cancel{|G_0|^2} + |G_1|^2 + |G_{-1}|^2 = 2|G_1|^2$

$G_1 = \pm \frac{j}{2} \rightarrow F_1 = \mp \frac{1}{2}$,

$f(t) = \mp \cos\frac{\pi}{2}t$

Fourier transform periodic → aperiodic ($\lim_{T\to\infty}$)

Def. Fourier integral: repeat ap. sig. at a freq. $T_0 \to f_{T_0}(t)$. $f(t)=\lim_{T_0\to\infty} f_{T_0}(t)$

$f_{T_0}(t)=\sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$, $\omega_0=\frac{2\pi}{T_0}$, $F_n=\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f_{T_0}(t)\, e^{-jn\omega_0 t}\, dt$

$\lim_{T_0\to\infty}$ → $\omega_0 \to 0$, call it $\Delta\omega$ $\quad =\frac{\Delta\omega}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{-jn\Delta\omega t}\, dt$

Let $\boxed{F(\omega)\equiv\int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt}$ $\quad =\frac{\Delta\omega}{2\pi}F(n\Delta\omega)$ → FT

$f_{T_0}(t)=\sum_{n=-\infty}^{\infty}\frac{\overset{\text{func. value}}{F(n\Delta\omega)}\,\Delta\omega}{2\pi}\, e^{jn\Delta\omega t}$

$\boxed{f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)\, e^{j\omega t}\, d\omega}$ → inverse FT (ana. F. series)

Spectrum (ω) is continuous

Existence – Weak cond. finite energy (power over inf. period) $\left(\int_{-\infty}^{\infty} f^2(t)\,dt<\infty\right)$ ⇒ finite $F(\omega)$ ($\int$ con.),

$\int_{-\infty}^{\infty} e^2(t)\,dt \to 0$

-Strong
- fin $\mathcal{E}$
- fin # of max/min w/i any fin. interval ⇒ Same
- fin # discont. (each discont is fin.)

E. $\mathcal{F}\{\delta(t)\}=\int_{-\infty}^{\infty}\delta(t)\, e^{-j\omega t}\, dt = 1$

narrow in time → wide in freq.

– $\mathcal{F}\{1\}=\int_{-\infty}^{\infty} 1\, e^{-j\omega t}\, dt \overset{?}{=} 2\pi\delta(\omega)$

calculated by inv. $\mathcal{F}^{-1}\{2\pi\delta(\omega)\}=\frac{1}{2\pi}2\pi\int --- = 1$

– $\mathcal{F}\{e^{-at}u(t),\ a>0\}=\int_0^{\infty} e^{-at}\, e^{-j\omega t}\, dt$

$$=\frac{-1}{a+j\omega}\left[e^{-(a+j\omega)t}\right]_0^{1}$$

complex spectrum $=\frac{1}{a+j\omega}$ $\qquad |F(\omega)|=\frac{1}{\sqrt{a^2+\omega^2}}$ $\qquad \measuredangle F(\omega)=-\tan^{-1}\frac{\omega}{a}$

$f(t)$ is real ⟶ $|F|$ even $\qquad \measuredangle F$ odd

– $\mathcal{F}\{e^{-a|t|},\ a>0\}=\int_{-\infty}^{\infty} e^{-a|t|}\, e^{-j\omega t}\, dt=\int_{-\infty}^{0} e^{(a-j\omega)t}\, dt+\int_0^{\infty} e^{-(a+j\omega)t}\, dt$

$$=\frac{1}{a-j\omega}+\frac{1}{a+j\omega}$$

$$=\frac{2a}{\omega^2+a^2}$$

$f(t)$ real, even ⟶ $F(\omega)$ real, even

E. window centered at 0 w/ width τ

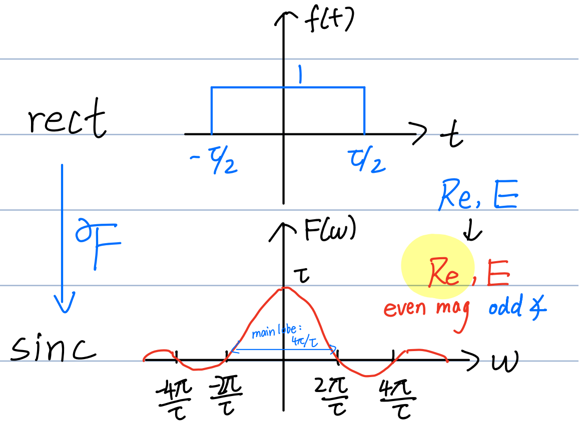
$$F(\omega)=\int_{-\tau/2}^{\tau/2} 1\cdot e^{-j\omega t}\,dt$$

$$=-\frac{1}{j\omega}\left[e^{-j\omega\tau/2}-e^{j\omega\tau/2}\right]$$

$$=\frac{2}{\omega}\sin\frac{\omega\tau}{2}$$

$$=\tau\,\frac{\sin\frac{\omega\tau}{2}}{\frac{\omega\tau}{2}}\equiv \mathrm{sinc}\left(\frac{\omega\tau}{2}\right),\ \mathrm{sinc}\ 0=1$$

$$=\tau\ \mathrm{sinc}\,\frac{\omega\tau}{2}$$

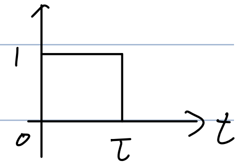


E. $F(\omega)=\begin{cases}1, & |\omega|\le W\\ 0, & |\omega|>W\end{cases} \leftrightarrow f(t)=\frac{W}{\pi}\,\mathrm{sinc}\,(Wt)$

$\mathcal{F}$ props. (series prop. carry over)

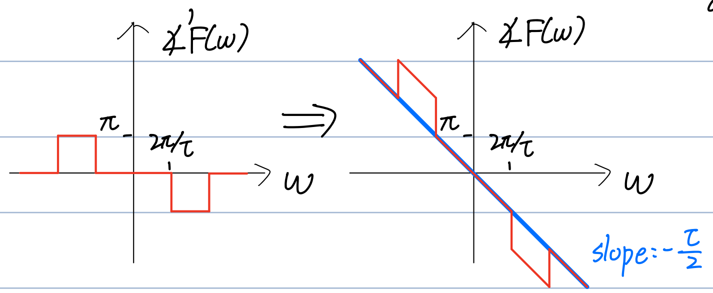
1. Linear $af(t)+g(t)\leftrightarrow aF(\omega)+G(\omega)$
2. Time shift $f(t-t_0)\leftrightarrow F(\omega)\,e^{-j\omega t_0}$ $\left(F_n\,e^{-jn(\underset{\omega}{\omega_0} t_0)}\right)$

$|F(\omega)|$ unchanged, but $\measuredangle F(\omega) -= \omega t_0$ → linear phase shift



E. $F(\omega)=\tau\ \mathrm{sinc}\,\frac{\omega\tau}{2}\,e^{-j\frac{\omega\tau}{2}}$ (shifted window)

$\measuredangle F(\omega) = \underset{\sim}{\measuredangle F'(\omega)} - \frac{\tau}{2}\omega$

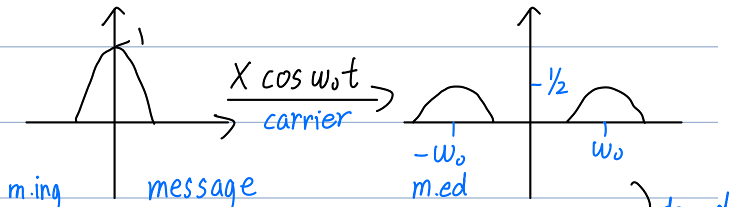


3. Frequency shift $f(t)e^{j\omega_0 t}\longleftrightarrow F(\omega-\omega_0)$ Pf. $\int$

E. $\mathcal{F}[f(t)\cos\omega_0 t]$

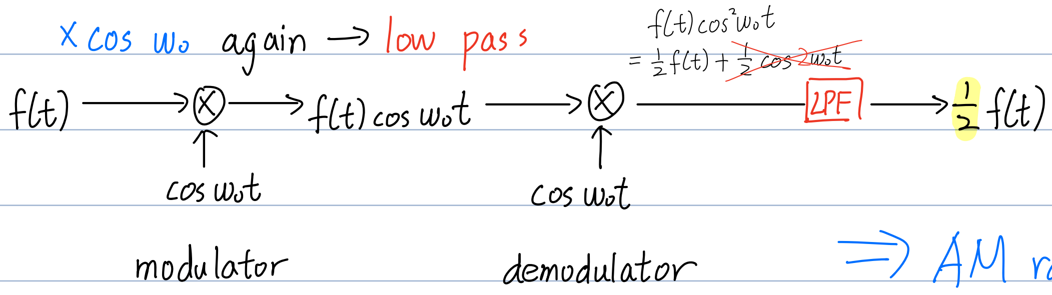
$$=\frac{1}{2}\left[\mathcal{F}[f(t)e^{j\omega_0 t}]+\mathcal{F}[f(t)e^{-j\omega_0 t}]\right]$$

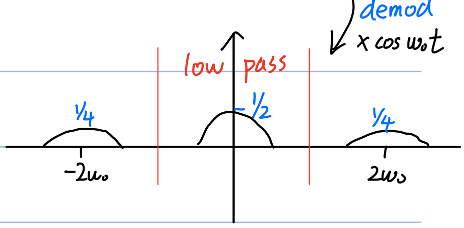
$$=\frac{1}{2}\left[F(\omega-\omega_0)+F(\omega+\omega_0)\right]$$



E. voice signal modulated to high ω_0, less trans. loss

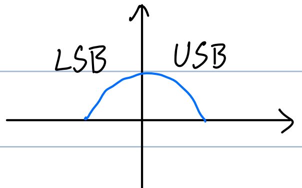
$\times\cos\omega_0$ again → low pass





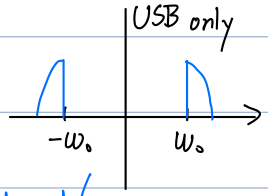
⇒ AM radio

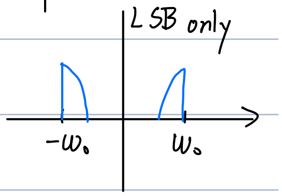
real-world signals: spectrum always symmetric



Only need half for information.

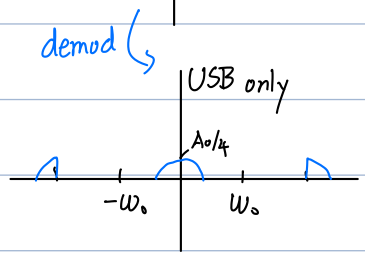
$\overset{\text{double side band}}{\text{DSB}} - \overset{\text{suppressed carrier}}{\text{SC}}$ (sinu. info. hidden) is wasteful





More efficient: SSB-SC (only half bandwidth)

mod-demod phase need to be complete sync.



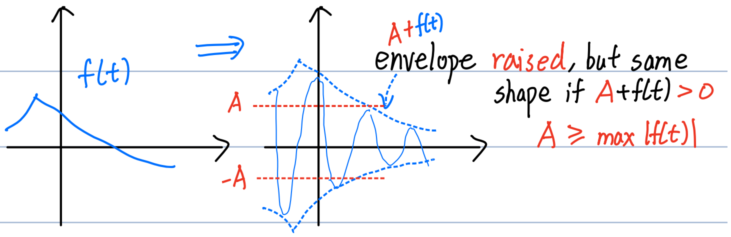
↓

Amplitude Modulation

$$\varphi_{AM}(t) = [A + f(t)]\cos\omega_0 t$$

$$= \underbrace{A\cos\omega_0 t}_{\text{carrier}} + \underbrace{f(t)\cos\omega_0 t}_{\text{DSB-SC}}$$

E.



envelope raised, but same shape if $A+f(t)>0$

$A \geq \max|f(t)|$

→ No need for receiver, just detect env for demod, cheap!

Problem: A needs to be large, power ineff. (E. broadcast, 1 trans ⇉ multi. receiver)

4. Time-freq duality

$f(t) \Leftrightarrow F(\omega)$

$F(t) \Leftrightarrow 2\pi f(-\omega)$

E. $\mathcal{F}[\frac{1}{a+jt}] = 2\pi e^{a\omega}u(-\omega)$; $\mathcal{F}[\frac{2a}{t^2+a^2}] = 2\pi e^{-a|\omega|}$

Pf. $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)e^{j\lambda t}\,d\lambda$ $(\omega \to \lambda)$

$2\pi f(-t) = \int_{-\infty}^{\infty} F(\lambda)e^{-jt\lambda}\,d\lambda = \mathcal{F}[F(\lambda)]$ $(t \to -\omega)$

5. Scaling

$f(at) \leftrightarrow \frac{1}{|a|}F(\frac{\omega}{a})$

Compress in t → expand in ω

Pf. 1. $a>0$ Let $\tau = at$, $\int$ — — —

2. $a<0$ — — — $\int_{\infty}^{-\infty}$ — — —

6. Reversal (t/ω)

$f(t) \Leftrightarrow F(\omega)$

$f(-t) \Leftrightarrow F(-\omega)$

Simple.

7. Convolution

$f(t) * g(t) \leftrightarrow F(\omega) G(\omega)$

$f(t)\, g(t) \leftrightarrow \frac{1}{2\pi} F(\omega) * G(\omega)$

Pf. $\mathcal{F}[f*g]$

$= \int_{-\infty}^{\infty} e^{-j\omega t} \int_{-\infty}^{\infty} f(\tau) g(t-\tau)\, d\tau\, dt$

$= \int_{-\infty}^{\infty} f(\tau) \int_{-\infty}^{\infty} g(t-\tau) e^{-j\omega t}\, dt\, d\tau$

$= \int_{-\infty}^{\infty} f(\tau)\, \mathcal{F}[g(t-\tau)]\, d\tau$

$= \int_{-\infty}^{\infty} f(\tau)\, G(\omega) e^{-j\omega\tau}\, d\tau$

$= G(\omega)\, F(\omega)$

$e^{-j\omega t} \rightarrow e^{+jt\omega}$

$\mathcal{F}^{-1}\left[\frac{1}{2\pi} F(\omega) * G(\omega)\right]$

$= \left(\frac{1}{2\pi}\right)^2 \int e^{j\omega t} \int F(\lambda) G(\omega-\lambda)\, d\lambda\, d\omega$

$= \frac{1}{2\pi} \int F(\lambda) \left[\frac{1}{2\pi} \int G(\omega-\lambda) e^{jt\omega}\, d\omega\right] d\lambda$

$= \frac{1}{2\pi} \int F(\lambda)\, \mathcal{F}^{-1}[G(\omega-\lambda)]\, d\lambda$

$= \frac{1}{2\pi} \int F(\lambda)\, g(t)\, e^{jt\lambda}\, d\lambda$

$= g(t)\, f(t)$

8. Time differentiation

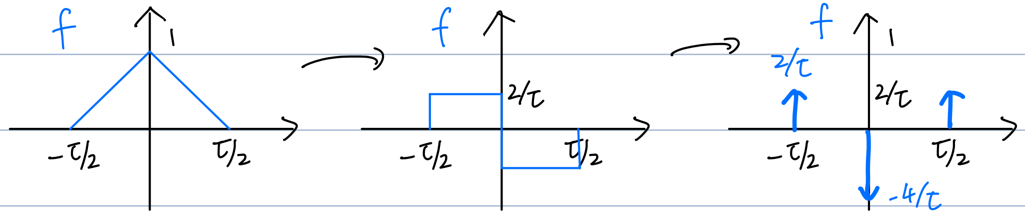
$\frac{df(t)}{dt} \leftrightarrow (j\omega) F(\omega)$ diff → multi.

$\frac{d^n f(t)}{dt^n} \leftrightarrow (j\omega)^n F(\omega)$

Pf. $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t}\, d\omega$

$\dot{f}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) (j\omega) e^{j\omega t}\, d\omega = \mathcal{F}^{-1}[(j\omega) F(\omega)]$

E.



$\mathcal{F}[\ddot{f}(t)] = \frac{2}{\tau}\left[e^{j\frac{\tau}{2}\omega} + e^{-j\frac{\tau}{2}\omega}\right] - \frac{4}{\tau}$

$= \frac{2}{\tau} \cdot 2\cos\left(\frac{\omega\tau}{2}\right) - \frac{4}{\tau}$

$\frac{d^2}{dt^2} \Big\downarrow$

$= \frac{4}{\tau}\left(\cos\frac{\omega\tau}{2} - 1\right)$

$(j\omega)^2 F(\omega) = -\frac{8}{\tau} \sin^2\frac{\omega\tau}{4}$

$F(\omega) = \frac{8}{\omega^2\tau} \sin^2\frac{\omega\tau}{4}$

$= \underline{\frac{\tau}{2}\, \mathrm{sinc}^2 \frac{\omega\tau}{4}} \rightarrow 2$ rect convolved (rect w/ width $\frac{\tau}{2}$, amp. $\sqrt{\frac{2}{\tau}}$)

$\frac{2}{\tau}\, f(2t) * f(2t) \leftrightarrow \frac{2}{\tau}\, \frac{1}{2} F\left(\frac{\omega}{2}\right) \cdot \frac{1}{2} F\left(\frac{\omega}{2}\right) = \frac{\tau}{2}\, \mathrm{sinc}^2\left(\frac{\omega\tau}{4}\right)$

9. Time integration $\int_{-\infty}^{t} f(\tau)\,d\tau \leftrightarrow \frac{1}{jw}F(w) + \pi F(0)\delta(w)$

Pf. $\int_{-\infty}^{t} f(\tau)\,d\tau = \int_{-\infty}^{\infty} f(\tau)\,u(t-\tau)\,d\tau = f(t) * u(t) \leftrightarrow F(w)\,U(w)$

$$U(w) = \int_{-\infty}^{\infty} u(t) e^{-jwt}\,dt \qquad u(t) = \lim_{a\to 0} e^{-at}\,u(t)$$

$$= \frac{1}{-jw}\, e^{-jwt}\Big|_{0}^{\infty} \qquad U(w) = \lim_{a\to 0} \frac{1}{a+jw}$$

diverges → must have δ

$$= \lim_{a\to 0}\left(\frac{a}{a^2+w^2} - j\frac{w}{\cancel{a^2}+w^2}\right)$$

$$\begin{Bmatrix} w\neq 0, & 0 \\ w=0, & \pi\delta! \end{Bmatrix} \longrightarrow \int_{-\infty}^{\infty} \frac{a}{w^2+a^2}\,dw = \pi \rightarrow = \boxed{\pi\,\delta(w) + \frac{1}{jw}}$$

10. Conjugation $f(t)^* \leftrightarrow F(-w)^*$ $\quad (f(t)^* \leftrightarrow F_{-n}^*)$

Pf. $\mathcal{F}[f^*] = \int f^* e^{-jwt}\,dt = \left[\int f\, e^{-j(-w)t}\,dt\right]^*$

11. Sym a) $f(t)$ real → $F(-w) = F(w)^*$ (mag. even, angle odd)

b) $f(t)$ real, even → F real, even

c) $f(t)$ real, odd → F im, odd

d) $\mathcal{F}[f_e(t)] = \frac{1}{2}(F(w) + F(-w)) = \frac{1}{2}(F(w) + F(w)^*) = Re\{F(w)\}$

e) $\mathcal{F}[f_o(t)] =$ ________________ $= Im\{F(w)\}$

Same as FS

12. Parseval $E_f = \int |f(t)|^2\,dt = \frac{1}{2\pi}\int |F(w)|^2\,dw$ (for $\overset{\text{FS: power}}{\text{energy}}$ sig.)

Pf. $= \int f(t)\cdot f(t)^*\,dt$

$= \int f(t)\; \mathcal{F}^{-1}\{F(-w)^*\}\,dt$

$= \int f(t)\,\frac{1}{2\pi}\int F(-w)^* e^{jwt}\,dw\,dt$

(u-sub) $= \frac{1}{2\pi}\int f(t)\int_{\cancel{\infty}}^{\cancel{-\infty}} \cancel{-}F(\lambda)^* e^{-jt\lambda}\,d\lambda\,dt$

$= \frac{1}{2\pi}\int F(\lambda)^* \int f(t)\, e^{-j\lambda t}\,dt\,d\lambda$

$= \frac{1}{2\pi}\int F(\lambda)^* F(\lambda)\,d\lambda$

$= \frac{1}{2\pi}\int |F(w)|^2\,dw$

Energy spectra density : even for real f $\quad (|F(w)|^2 = F(w)\,F(w)^* = F(w)F(-w))$

E. real, E between w_1, w_2 : $\frac{\cancel{2}}{\cancel{2\pi}}\int_{w_1}^{w_2} |F(w)|^2\,dw$ $\quad$ (+ & −, so 2)

Essential bandwidth w band that contains certain % (E. 95) of signal energy

E. $f(t) = e^{-at}u(t)$

$F(\omega) = \frac{1}{a+j\omega}$

$|F(\omega)|^2 = \frac{1}{\omega^2+a^2} \rightarrow \frac{1}{\pi}\int_0^W \frac{1}{\omega^2+a^2}d\omega = 0.95\int_0^\infty \frac{1}{\omega^2+a^2}d\omega$

$\frac{1}{a\pi}\tan^{-1}\left(\frac{\omega}{a}\right)\Big|_0^W = 0.95\cdot\frac{1}{2a}$

$W \approx 12.71\,a$ rad/s $\approx 2a$ Hz

Q. $\mathcal{F}^{-1}[|F(\omega)|^2] = \mathcal{F}^{-1}[F(\omega)\cdot F(\omega)^*] = f(t) * f(-t)^*$

$= \int f(\tau)f^*(\tau-t)\,d\tau \equiv \psi_f(t)$

Autocorrelation of f(t) (delay → *, corr. w/ time change)

System transmission (LTI) $y_{zs} = f * h$

aka filter $Y_{zs}(\omega) = F(\omega)H(\omega) \xrightarrow{\text{def.}}$ freq. response

$|Y_{zs}| = |F|\,|H| \longrightarrow$ mag. res.

$\angle Y_{zs} = \angle F + \angle H \rightarrow$ angle res.

Distortionless trans. $y(t) = kf(t-t_0)$ (just delay)

$h(t) = k\,\delta(t-t_0)$

$H(\omega) = k\cdot 1\,e^{-j\omega t_0} \rightarrow$ mag. const; ∡. linear

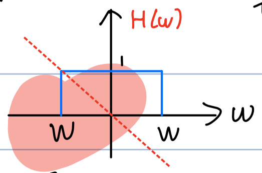
E. ear: more sensitive to mag. distortion, not to ∡

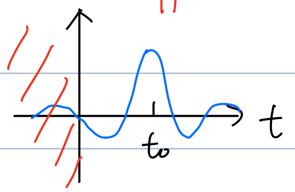
eye: ✓ to ∡ , X to mag

Ideal filter Distortionless LTI system → $|H| = \overset{\text{const.}}{k}$, $\angle H = \overset{\text{lin}}{-t_0\omega}$

↳ Only need at ω freq. of interest → suppress all other bands

E. ideal LPF (f<B)





$h(t) = \frac{W}{\pi}\,\text{sinc}[W(t-t_0)]$

- h(τ) non-causal $y = f*h = \int h(\tau)f(t-\tau)\,d\tau$

↳ non-c system!

$= \int_{-\infty}^{0} h(\tau)\overbrace{f(t-\tau)}^{\tau<0,\text{ future input}}d\tau + \int_{0}^{\infty} h(\tau)\overbrace{f(t-\tau)}^{\text{past input}}d\tau$

Physically unrealizable! (also high-pass / band-pass)

Paley-Wiener Criterion: $H(\omega)$ is realizable ($h(t)$ is causal) iff $\int_{-\infty}^{\infty} \frac{|\ln|H(\omega)||}{1+\omega^2} d\omega < \infty$

E. ideal filters has consecutive 0-intervals, $\ln \to \infty \Rightarrow \int \to \infty$

Practical ~ 1. Truncate ideal's non-causal part $\to \hat{h}(t) = h(t)u(t)$

$$\hat{H}(\omega) = H(\omega) * U(\omega)$$

Periodic FT $\quad f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \qquad 1\cdot e^{j\omega_0 t} \longleftrightarrow 2\pi\,\delta(\omega-\omega_0)$

$$\mathcal{F}[f(t)] = \sum F_n\, 2\pi\,\delta(\omega-n\omega_0)$$

$$= 2\pi \sum F_n\, \delta(\omega-n\omega_0)$$

E. $\mathcal{F}\{\sin\omega_0 t\} = \mathcal{F}\{\frac{1}{2j}[e^{j\omega_0 t} - e^{-j\omega_0 t}]\} = \frac{\pi}{j}[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)]$

E. $\quad$ cos $\qquad \pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$

E. $\mathcal{F}\{\sum_{-\infty}^{\infty} \delta(t-nT_0)\} = 2\pi \cdot \frac{1}{T_0} \sum \delta(\omega-n\omega_0)$ $\qquad$ pulse train ↔ pulse train

LTI response to periodic sig.

$$F(\omega) = 2\pi \sum F_n\, \delta(\omega-n\omega_0)$$

$$Y(\omega) = F(\omega)H(\omega) = 2\pi \sum F_n H(\omega)\, \delta(\omega-n\omega_0)$$

$$= 2\pi \sum F_n H(n\omega_0)\, \delta(\omega-n\omega_0)$$

$$= 2\pi \sum Y_n\, \delta(\omega-n\omega_0), \quad \text{where } Y_n \equiv F_n \cdot H(n\omega_0)$$

$y(t)$ is periodic w/ the same ω_0 and F. coeff. $= F_n \cdot H(n\omega_0)$

LTI can't change input ω! $\qquad$ H is a filter (fucn. of ω)

E1. $\quad F_1 = 1,\ F_n = 0$ else $\qquad (f(t) = e^{j\omega_0 t})$

$Y_1 = H(\omega_0) \longrightarrow y(t) = H(\omega_0)\, e^{j\omega_0 t}$

scaled but same → eigenfunction! if $y = \lambda f$

E2. $f(t) = \cos(\omega_0 t + \theta)$, assume $h(t)$ real $\to |H|$ even, $\measuredangle H$ odd

$$= \frac{1}{2} e^{j\theta} e^{j\omega_0 t} + \frac{1}{2} e^{-j\theta} e^{-j\omega_0 t}$$

$$y(t) = \frac{1}{2} e^{j\theta} H(\omega_0)\, e^{j\omega_0 t} + \frac{1}{2} e^{-j\theta} H(-\omega_0)\, e^{-j\omega_0 t}$$

$$= \frac{1}{2} e^{j\theta} |H(\omega_0)|\, e^{j\measuredangle H(\omega_0)} e^{j\omega_0 t} + \frac{1}{2} e^{-j\theta} |H(\omega_0)|\, e^{-j\measuredangle H(\omega_0)} e^{-j\omega_0 t}$$

$= |H(\omega_0)| \cdot \frac{1}{2}\left[e^{j(\omega_0 t+\theta+\measuredangle H(\omega_0))} + e^{-j(\omega_0 t+\theta+\measuredangle H(\omega_0))}\right]$

$= |H(\omega_0)| \cos(\omega_0 t+\theta+\measuredangle H(\omega_0))$ (same for sin)

E. Convolution $\underbrace{\cos 2t}_{f(t)} * \underbrace{e^{-3t} u(t)}_{h(t)} \longrightarrow H(\omega) = \frac{1}{3+j\omega}$

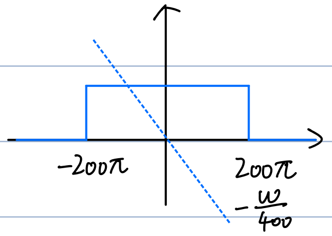
$H(\omega_0) = H(2)$, $|H| = \frac{1}{\sqrt{13}}$, $\measuredangle H = -\tan^{-1}\frac{2}{3}$

$* = \frac{1}{\sqrt{13}} \cos(2t - \tan^{-1}\frac{2}{3})$

E. $f(t) = \cos(50\pi t + \frac{\pi}{3}) + \sin(250\pi t + \frac{\pi}{3}) + e^{j80\pi t}$

H(ω) is an ideal LPF

$y(t) = 1\cdot\cos(50\pi t + \frac{\pi}{3} - \frac{50\pi}{400}) + 0 + 1\, e^{j(-\frac{80\pi}{400})} \cdot e^{j80\pi t}$



11/7

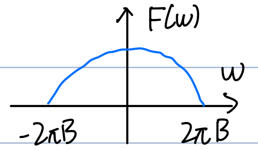
Sampling Take sig values every Δ seconds → discrete time

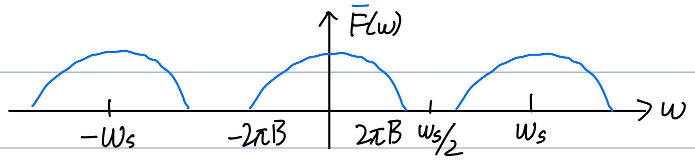
? can discrete sample $\xrightarrow{\text{recover}}$ cont. time signal? Not quite for t-domain

$\bar{f}(t) = f(t)\cdot\delta_{T_s}(t) \longleftrightarrow \bar{F}(\omega) = \cancel{\frac{1}{2\pi}} F(\omega) * \left[\frac{\cancel{2\pi}}{T_s}\sum_{n=-\infty}^{\infty}\delta(\omega - n\omega_s)\right]$

$= \sum_{n=-\infty}^{\infty} f(nT_s)\,\delta(t-nT_s)$ $= \frac{1}{T_s}\sum_{n=-\infty}^{\infty} F(\omega - n\omega_s)$ shift → add

ω-domain

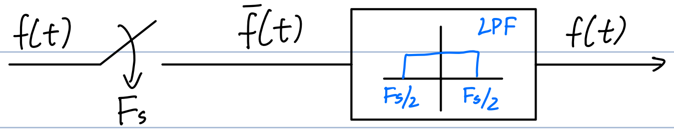




For recovery via LPF, $\omega_s \geq 4\pi B$

$F_s \geq 2B$

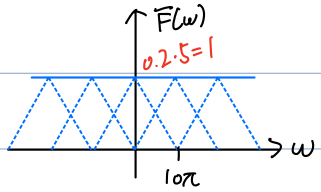
Thm. (Nyquist sampling) If f(t) band-limited to B Hz, it can be perfectly recovered from samples taken at a rate $F_s > \overset{\text{Nyquist rate}, N_B}{2B}$ samples/s



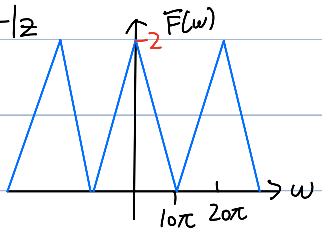
E. $f(t) = \text{sinc}(100\pi t) \to B = 100\pi = 50$ Hz, $F_N = 2B = 100$ Hz

E. $f(t) = \text{sinc}^2(5\pi t) \longleftrightarrow 0.2\Delta(\frac{\omega}{20\pi}) \to B = 5$Hz, $F_N = 10$Hz

If $F_s = \overset{10\pi}{5\text{Hz}}$, amp ×5



$F_s = \overset{20\pi}{10\text{Hz}}$



Reconstruct $F(\omega) = \bar{F}(\omega) \cdot \overset{\text{amp}}{\frac{1}{F_s}} \overset{\text{LPF}}{\text{rect}}\left(\frac{\omega}{4\pi B}\right)$ if $T_s = \frac{1}{2B} = \frac{1}{F_N}$

if $F_s = F_N = 2B$ $f(t) = \bar{f}(t) * \cancel{\frac{2B}{F_s}} \text{sinc}(2\pi B t)$

$$= \sum_{n=-\infty}^{\infty} f(nT_s)\,\delta(t-nT_s) * \text{sinc}(2\pi B t)$$

$$= \sum f(nT_s)\,\text{sinc}(2\pi B(t-nT_s))$$

$T_s = \frac{1}{F_s} = \frac{1}{2B}$ $= \sum f(nT_s)\,\text{sinc}(2\pi B t - n\pi)$ → linear comb. of basis! ana FS

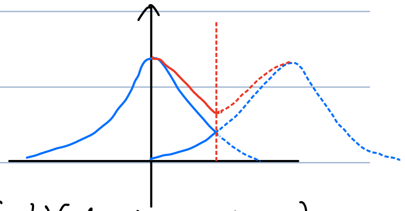
interpolation formula

General $F(\omega) = \bar{F}(\omega)W(\omega)$ W is a relaxed filter

$(F_s > 2B)$ $f = \bar{f} * w(t)$

$= \sum_n f(nT_s)\,w(t-nT_s)$ not sinc

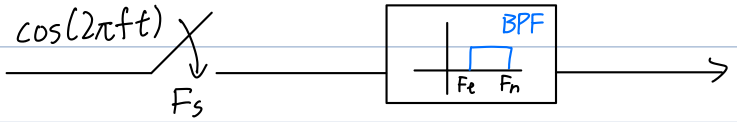
Real can't be $\overset{\checkmark}{\text{t-lim}}$ and $\overset{\times}{\omega\text{-lim}}$ → sample below F_N → distort :(



due to 1. All hi-ω parts cut by LPF
add 2. folded back to lo-ω part $\xrightarrow{\text{def.}}$ Aliasing

Solution (anti-aliasing filter) cut tail via LPF ($F_s/2$, forces N_q); 2 ✓, 1 X (don't care hi-ω)

E.



Need $\exists n$ $F_\ell < \overset{\text{sinusoid}}{\pm f} + nF_s < F_h$

E. $f = \pm 30\,\text{Hz}$, $F_s = 50\,\text{Hz}$, $F_\ell = 0$, $F_h = F_s/2 = 25\,\text{Hz}$ ⇒ $n = 1$ so $f = -30\,\text{Hz}$ ✓