Time-domain Analysis

Complex # j s.t. $j^2 = -1$

$z = a + jb = re^{j\theta}$

$z^* = a - jb = re^{-j\theta}$

u(t) appendix

abs sometimes / square $\not A$

don't forget $\delta(t)$ when $'$, $\int$

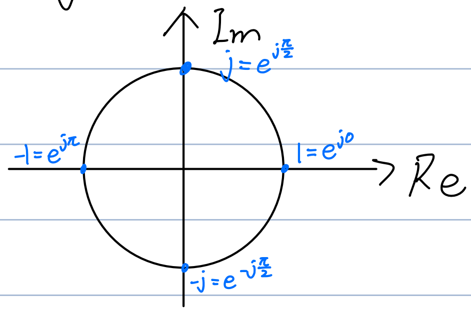
0^+ and 0^- !

don't forget T_0 in $\mathcal{E}_i$

don't forget δ initially and finally

graphical: $\int \frac{f(t)}{g(t)}$! may not just t!

Special cmplx #



$\theta = \tan^{-1}\left(\frac{b}{a}\right) \rightarrow$ only gives Q1, Q4 May need $\pm\pi$

First look a, b signs

$$z^{1/n} = r^{1/n} e^{j\frac{\theta + 2\pi m}{n}} \qquad 0 \le m < n$$

Sinusoid $f(t) = C\cos(\underbrace{2\pi \overset{Hz}{F_0}}_{\omega_0} t + \theta)$

Add $C\cos(\omega_0 t + \theta) = C\cos(\omega_0 t)\cos(\theta) - C\sin(\omega_0 t)\sin(\theta)$ Same ω_0!

$= a\cos(\omega_0 t) + b\sin(\omega_0 t)$

where $\left.\begin{array}{l} a = C\cos\theta \\ b = -C\sin\theta \end{array}\right\}$ $\quad C = \sqrt{a^2 + b^2}$

$\theta = \tan^{-1} -\frac{b}{a} \; (\pm\pi)$ $\qquad \sin + \cos \xrightarrow{\text{ana. cmplx}} \cos(\sim + \theta)$

E. $f(t) = \cos(\omega_0 t) - \sqrt{3}\sin(\omega_0 t)$ $\qquad a = 1,\ b = -\sqrt{3}$

$C = \sqrt{a^2 + b^2} = 2, \quad \theta = \frac{\pi}{3}$

$= 2\cos(\omega_0 t + \frac{\pi}{3})$

Size of signal

Signal a function of some i.v. (E. time, space)

System a sig. processor $\xrightarrow{i} \square \xrightarrow{o}$

Signal energy $E_f = \int_{-\infty}^{\infty} f^2(t)\,dt$ (if $f(t)$ is real)

$\hookrightarrow = \int_{-\infty}^{\infty} |f(t)|^2\,dt$ (complex) $E_{Re} + E_{Im}$

~ power $P_f = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |f(t)|^2\,dt$ avg.

rms power $\sqrt{P_f}$

If $f(t)$ periodic, $P_f = \frac{1}{T}$ ~

Classification

1. Time – continuous (real) vs discrete (computer)
2. $f(t)$ analog vs digital
3. periodic ($\exists T > 0$ s.t. $f(t) = f(t+T)$) vs aperiodic
   - $\hookrightarrow$ can be generated by periodic extension of segments of T.
4. causal ($f(t) = 0$ when $t < 0$) vs noncausal vs anticausal (~ $t > 0$)
5. energy ($E_f < \infty$ $(P_f = 0)$) vs power ($0 < P_f < \infty$ $(E_f = \infty)$) both X, neither ✓
6. deterministic (For any t, we know $f(t)$) vs stochastic (known probabilistically) (random)
   - deterministic $\downarrow$ no info
   - stochastic $\downarrow$ info-carrying

Operations

1. time shift $f(t) \to \phi(t) = f(t-\tau)$
2. ~ scaling $f(t) \to \phi(t) = f(at)$ $(a > 0)$
3. ~ reversal $f(t) \to \phi(t) = f(-t)$

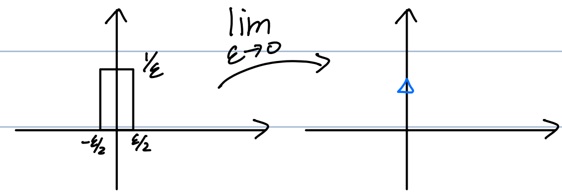
Useful signals 1. Unit step $u(t) \to$ causal $\to$ Any causal can be in $f(t)u(t)$

〜 anti 〜〜〜 $f(t)u(-t)$

Window : $u(t-a) - u(t-b)$, $a \le b$

2. impulse $\delta(t) = 0$; $t \neq 0$

$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$



Prop. 1. If $f(t)$ is cont. at t_0, $f(t)\,\delta(t-t_0) = f(t_0)\,\delta(t-t_0)$

2. 〜〜〜〜〜〜 $\int_{-\infty}^{\infty} f(t)\,\delta(t-t_0) = f(t_0)$

3. $\delta(t) = \frac{du(t)}{dt}$, $\dot{u}(t) = \delta(t)$

3. Complex exp $f(t) = e^{st}$, where $s = \sigma + j\omega$ (complex freq.)

$= e^{\sigma t} e^{j\omega t}$

$= e^{\sigma t}(\cos(\omega t) + j\sin(\omega t))$

a) $\omega = 0$ $f(t) = e^{\sigma t}$ (exp)

b) $\sigma = 0$ $f(t) = \cos(\omega t) + j\sin(\omega t)$ (pure sine)

c) $\sigma < 0$, $\omega \neq 0$

d) $\sigma > 0$, $\omega \neq 0$

4. Even $f_e(-t) = f_e(t)$ $\int_{-a}^{a} f_e(t)\,dt = 2\int_0^a f_e(t)\,dt$

Odd $f_o(-t) = -f_o(t)$ 〜〜〜 $= 0$

☆ For any $f(t)$, can be written as $f_e(t) + f_o(t)$

where $f_e(t) = \frac{1}{2}(f(t) + f(-t))$

$f_o(t) = \frac{1}{2}(f(t) - f(-t))$

E. $f(t) = e^{-2t}u(t)$

E. $f(t) = \begin{cases} f_0(t) & t < t_0 \\ f_1(t) & t > t_0 \end{cases}$

$f(t) = f_0(t)\,u(-t+t_0) + f_1(t)\,u(t-t_0)$

$f'(t) = f_0'(t)\,u(-t+t_0) + f_0(t)(-1)\,\delta(-t+t_0)$

$+ f_1'(t)\,u(t-t_0) + f_1(t)\,\delta(t-t_0)$

$= [f_0'(t)\,u(-t+t_0) + f_1'(t)\,u(t-t_0)] + (f_1(t_0) - f_0(t_0))\,\delta(t-t_0)$ (even)

Systems $f(t) \rightarrow$ sys, $\mathcal{T}$ (black box) $\xrightarrow{y(t)}$ $\mathcal{T}[f(t)] = y(t)$

LTI systems

① Linear / non ~ iff scaling & additive aka superposition

$\mathcal{T}[kf(t)] = ky(t)$ $\quad \mathcal{T}[f_1+f_2] = y_1+y_2$

Let $D^n \equiv \frac{d^n}{dt^n}$, $\sum_{k=0} a_k D^k y(t) = \sum_{\ell=0} b_\ell D^\ell f(t)$, lin. if a_k, b_ℓ are not functions of $y(t), f(t)$

E. $\dot{y} + t^2 y = (2t+3)y$ is linear!

② Time var./invar. invar if $\mathcal{T}[f(t-\tau)] = y(t-\tau)$ E. $y = \sin t \; f(t-2)$

?
1. Let $g(t) \equiv f(t-\tau)$,
2. Compute $z(t) = \mathcal{T}[g(t)]$
3. Compute $y(t-\tau)$
4. Check if 2=3.

1. $g(t) = f(t-\tau)$
2. $z(t) = \sin t \; g(t-2) = \sin t \; f(t-\tau-2)$
3. $y(t-\tau) = \sin(t-\tau) f(t-\tau-2)$
4. ✗

Must be const. coeff., or coeff. are func. of $f(t), y(t)$.

coeff. are:
① All const. $\rightarrow$ LTI
② indep. fucn. of t, but not $f(t), y(t)$ $\rightarrow$ L
③ fucn. of $f(t), y(t)$, not indep. of t $\rightarrow$ TI

③ Instantaneous vs Dynamic

Instantaneous: $y(t)$ only depends on $f(t)$, not past/future (no memory)

Dynamic: not ~

E. Sys. described by DFQ $\rightarrow$ need $\int$, dynamic

④ Causal (system) vs Noncausal

real world, not sig.

Causal: $y(t)$ depends only on $f(\tau)$ $\tau \le t$ (current & past)

Noncausal: ~

⑤ Invertible: Can get $f(t)$, given $y(t)$

Non ~: ~ E. $y(t) = |f(t)|$ ✗

Convolution $\quad (f_1 * f_2)(t) = \int_{-\infty}^{\infty} \underbrace{f_1(\tau) f_2(t-\tau)}^{\text{shifted \& reflected}} \, d\tau$ (depends on the whole sig)

Prop. 1. commutativity $\quad f_1 * f_2 = f_2 * f_1$

Pf. Let $u = t - \tau$, ~

2. distributivity $\quad f_1 * (f_2 + f_3) = f_1 * f_2 + f_1 * f_3$ $\quad$ Pf. ~

3. associativity $\quad f_1 * (f_2 * f_3) = (f_1 * f_2) * f_3$

Pf. Let $g = f_2 * f_3 = f_3 * f_2 = \int_{-\infty}^{\infty} f_3(\tau_1) f_2(t-\tau_1)\, d\tau_1$

$$f_1 * g = \int f_1(\tau_2)\, g(t-\tau_2)\, d\tau_2$$
$$= \int f_1(\tau_2) \left[\int f_3(\tau_1) f_2(t-\tau_2-\tau_1)\, d\tau_1\right] d\tau_2$$
$$= \int f_3(\tau_1) \int f_1(\tau_2) f_2(t-\tau_1-\tau_2)\, d\tau_2\, d\tau_1 = f_3 * (f_1 * f_2) \checkmark$$

4. shift $\quad$ if $f_1 * f_2 = g$, then $f_1(t-\tau_1) * f_2(t-\tau_2) = g(t-\tau_1-\tau_2)$ $\quad$ Pf. evaluate

~ time-inv. prop.

5. impulse $\quad f * \delta = f$ $\quad$ Pf. ~

$$f(t) * \delta(t-t_0) = f(t-t_0)$$

6. width $\quad$ if f_1 defined $[T_0^{(1)}, T_1^{(1)}]$, f_2 defd. $[T_0^{(2)}, T_1^{(2)}]$,

$f_1 * f_2$ defd. $[T_0^{(1)} + T_0^{(2)}, T_1^{(1)} + T_1^{(2)}]$

Pf. $\quad f_1 * f_2 = \int \underbrace{f_1(\tau)}_{\tau \in [T_0^{(1)}, T_1^{(1)}]} \underbrace{f_2(t-\tau)}_{t-\tau \in [T_0^{(2)}, T_1^{(2)}]} d\tau \longrightarrow t \in [\sim + \sim, \sim + \sim]$

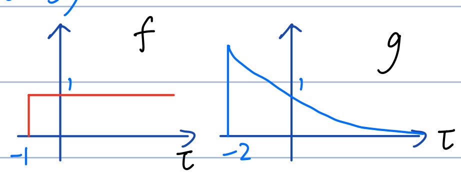
E. $\quad f_1 = e^{-t} u(t)$, $f_2 = e^{-2t} u(t-3)$

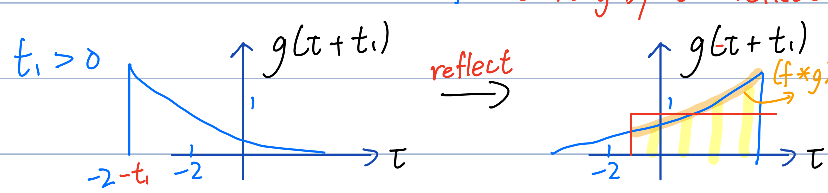
Range: $t > 3$ $\quad$, $f_1 * f_2 = \int_{-\infty}^{\infty} e^{-\tau} u(\tau)\, e^{-2(t-\tau)} u((t-\tau)-3)\, d\tau$

$$= \int_0^{t-3} e^{-\tau-2t+2\tau}\, d\tau$$
$$= \int_0^{t-3} e^{\tau-2t}\, d\tau$$
$$= e^{-2t}(e^{t-3} - 1) \quad u(t-3)$$

Graphical $\quad c = f * g = \int \underbrace{f(\tau)}_{f} \underbrace{g(t-\tau)}_{\text{shift } g \text{ by } t \to \text{reflect}} d\tau$

E.



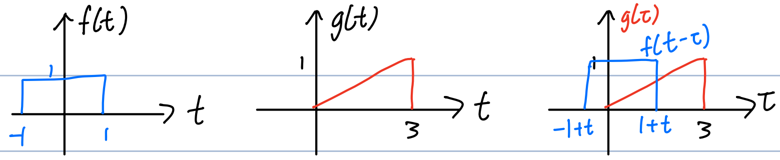


area under curve (overlap)

↳ if easy to compute, ✓

As $t_1 \uparrow$, more overlap

As $t_1 < -3$, no overlap, 0

E. 

Fix complex, move / reflect simple!

① $t+1 \geq 0$, $t-1 \leq 0$ (overlap) $f*g = \frac{1}{6}(t+1)^2 \quad [u(t+1)-u(t-1)]$

② $3 \geq t+1$, $t-1 \geq 0$ ( ) $= \frac{t}{2} \quad [u(t-1)-u(t-2)]$

③ $t+1 \geq 3$, $t-1 \leq 3$ ( ) $= \frac{1}{2}(4-t)(1+\frac{t-1}{3}) \quad [u(t-2)-u(t-4)]$

LTI system response linear differential sys.

$$(D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0)\, y(t) = (b_m D^m + \cdots + b_0)\, f(t)$$

(Polynomial) $\quad Q(D)\; y(t) = P(D)\; f(t)$

$\{a_i\}, \{b_j\}$ const. for LTI.

Typically $m \leq n$ (integrator). If differentiating, noise make it unstable

Zero-input response $f(t)=0$ Solely from init. cond.

$$Q(D)\; y_{zi}(t) = 0$$

Zero-state response Assume init cond. $=0$ Solely from input

$$Q(D)\; y_{zs}(t) = P(D)\; f(t) \quad \text{w/ init.} = 0$$

Total response $\quad y(t) = y_{zi}(t) + y_{zs}(t)$ Linear!

Recall $\quad f(t) = f(t) * \delta(t)$

$$= \int_{-\infty}^{\infty} f(\tau)\, \delta(t-\tau)\, d\tau$$

$$= \lim_{n\to\infty} \sum_n f(n\Delta\tau)\, \delta(t-n\Delta\tau)\, \Delta\tau \rightarrow \text{sum of delayed deltas}$$

$$y(t) = \mathcal{T}[f(t)]$$

$$= \mathcal{T}[\sum_n \overbrace{f(n\Delta\tau)}^{\text{const. of } t}\, \delta(t-n\Delta\tau)\, \Delta\tau]$$

$$\overset{\text{lin}}{=} \sum_n f(n\Delta\tau)\, \Delta\tau\; \mathcal{T}[\delta(t-n\Delta\tau)]$$

Let $h(t) = \mathcal{T}[\delta(t)] \rightarrow$ impulse response

$$\overset{\text{TI}}{=} \sum_n f(n\Delta\tau)\, \Delta\tau\; h(t-n\Delta\tau)$$

$$= \int f(\tau)\, h(t-\tau)\, d\tau$$

$$= f(t) * h(t) = y_{zs}(t)$$

Init cond. Assume input $f(t)$ is causal (starts at $t=0$)

init is condition imm. before $t=0$: $t=0^-$ E. $y(0^-), \dot{y}(0^-)$

and cond. ~ after $t=0$: $t=0^+$

z-input $y_{zi}(0^-) = y_{zi}(0^+)$ → continuous at 0

z-state $y_{zs}(0^-) = 0$ → generally discont.

} same for $y^{(n)}$

E. $h(t) = e^{-t}u(t)$, $f(t) = u(t)$, $y(0^+) = 0$, $\dot{y}(0^+) = 2$. Find y_{zs}, $\dot{y}_{zs}(0^+)$ and y_{zi}, $\dot{y}_{zi}(0^-)$

$y_{zs}(t) = f(t) * h(t)$

$= (1-e^{-t})u(t)$ $y_{zs}(0^+) = 0$, $\dot{y}_{zs}(0^+) = 1$

$\dot{y}_{zs}(t) = \cancel{(1-e^{-t})}\delta(t) + e^{-t}u(t)$ $y_{zi}(0^+) = 0$, $\dot{y}_{zi}(0^+) = 1$ $= \{y_{zi}, \dot{y}_{zi}\}(0^-)$