(a) Step Response

Step response derivation:

(a) $$-\frac{R_5}{R_6}x(t) + \frac{R_5}{R_7}v_1(t) - \frac{R_5}{R_8}\frac{R_3}{R_4}R_2C_2\dot{x} = R_1R_2C_1C_2\ddot{x}$$

$$-K_1x + K_2v_1 - K_3\tau\dot{x} = \tau^2\ddot{x}$$

$$\tau^2\ddot{x} + K_3\tau\dot{x} + K_1x = K_2v_1$$

$$\ddot{x} + \frac{K_3}{\tau_1}\dot{x} + \frac{K_1}{\tau_1\tau_2}x = \frac{K_2}{\tau_1\tau_2}V_1$$

$\omega_0 = \sqrt{\frac{K_1}{\tau_1\tau_2}} = \sqrt{\frac{1}{1}} = 1s^{-1}$ $\frac{K_3}{\tau_1} = 1s^{-1}$ $Q = \frac{\omega_0}{K_3/\tau_1} = 1 = 2\zeta \rightarrow \zeta = \frac{1}{2}$

$\alpha = \frac{1}{2}s^{-1}$ $\ddot{x} + \frac{\omega_0}{Q}\dot{x} + \omega_0^2x = 1\cdot v_1(t)$

$\zeta < 1 \rightarrow$ underdamped $\ddot{x} + \dot{x} + x = 1$

$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \frac{\sqrt{3}}{2}$ $x(t) = x_F(t) + x_n(t)$

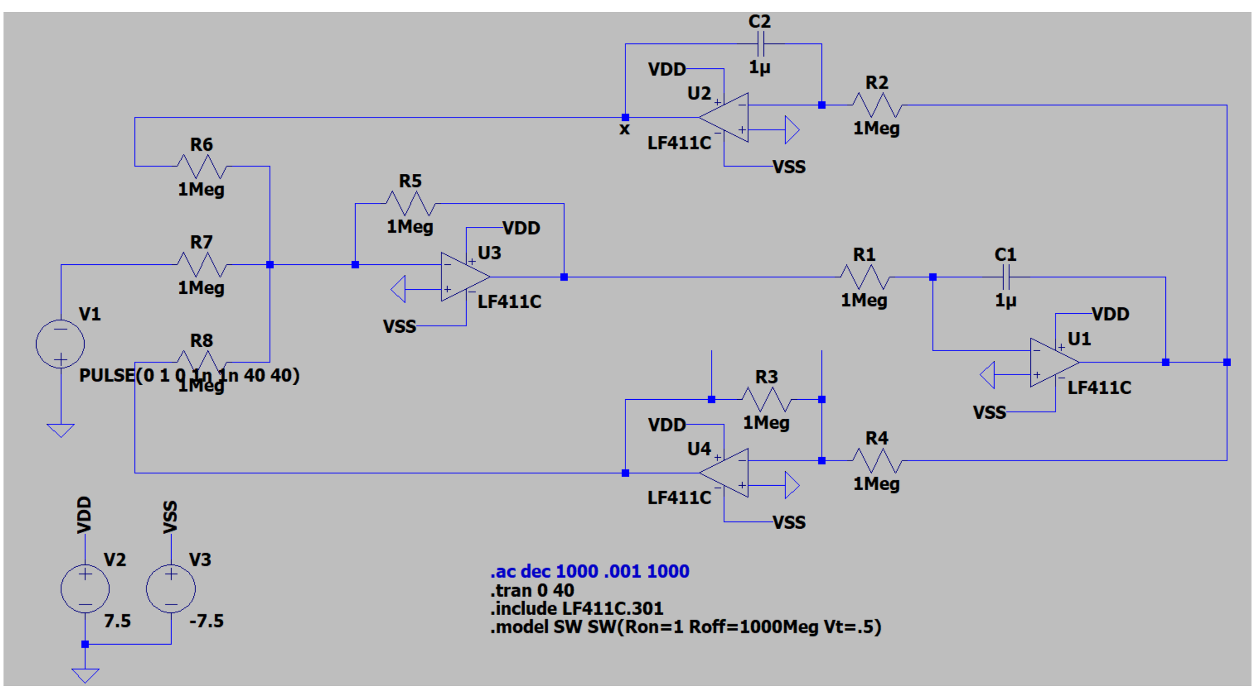
$x_F(t) = 1 \quad (v_1(t) = u(t))$ $= 1 + e^{-0.5t}\left(A\cos\frac{\sqrt{3}}{2}t + B\sin\frac{\sqrt{3}}{2}t\right)$

$x(0) = 0V$, $\dot{x}(0) = 0\ V/s$ $\Longrightarrow$ $A = -1$, $\frac{\sqrt{3}}{2}B - 0.5A = 0$

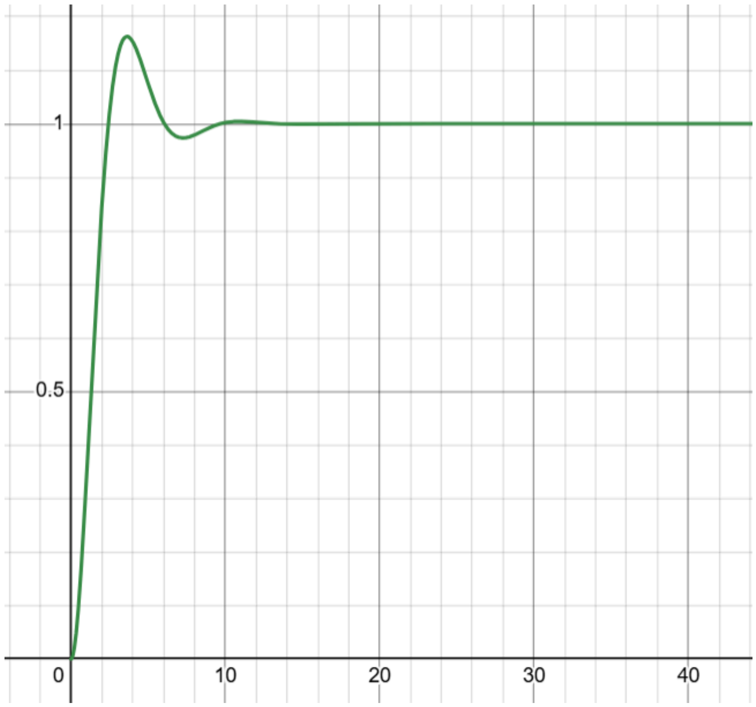
$B = \frac{1}{\sqrt{3}}$

Step response: $x(t) = 1 + e^{-0.5t}\left(-\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)\right)u(t)$

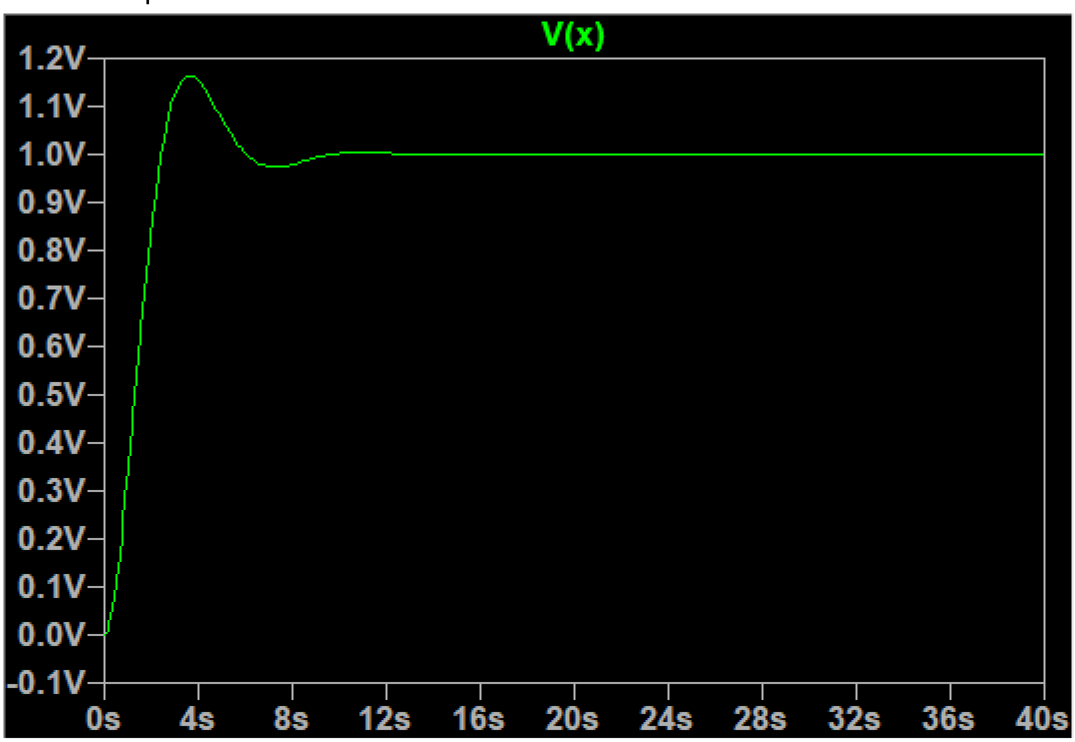
Circuit schematic:



Ideal plot:



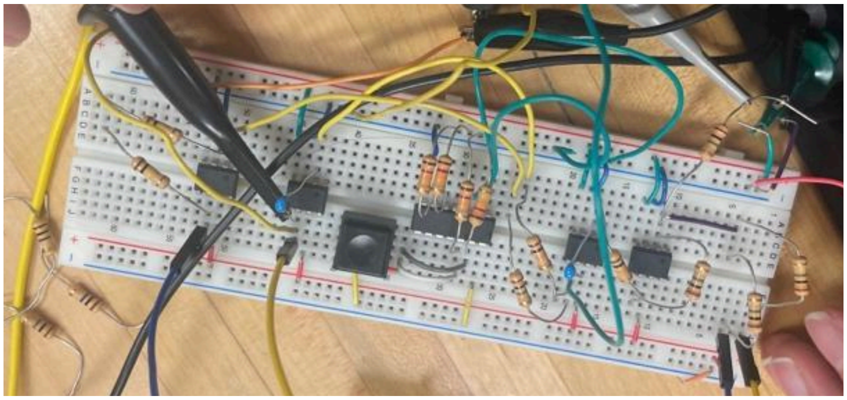
Simulation plot:



They look exactly the same :)

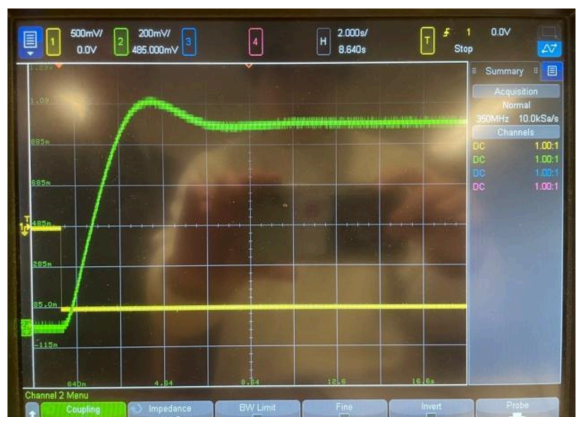
Physical Experiment

Circuit photo:



We used a CMOS switch array to ground the output of U2 and U4 to satisfy the initial conditions x(0) = 0 and x’(0) = 0.
We generated a square wave (yellow) with a long period to simulate the step input. The probe was on the negative terminal of the function generator, so the step input appeared negative.

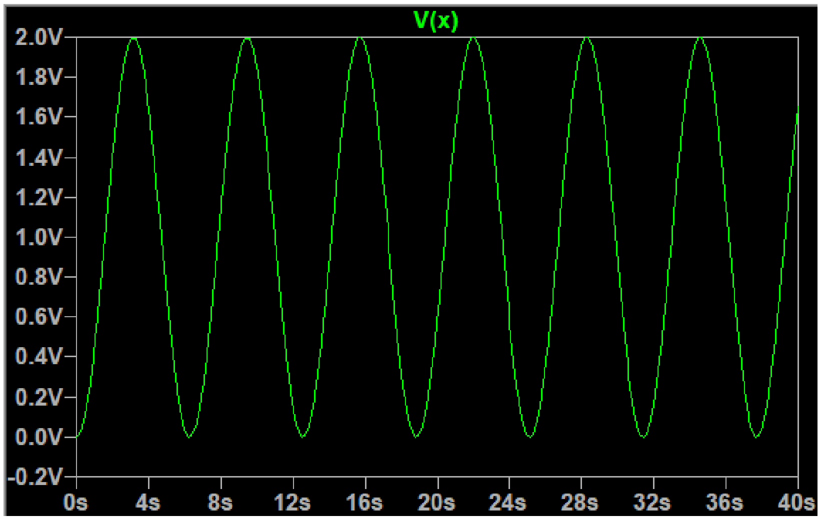
Below are the step response plots (green). The second figure is shifted and scaled for clarity.



They match perfectly with the theoretical and simulated behavior, with an overshooting peak at about 4s, followed by damped oscillation.

(b) Damping Factor

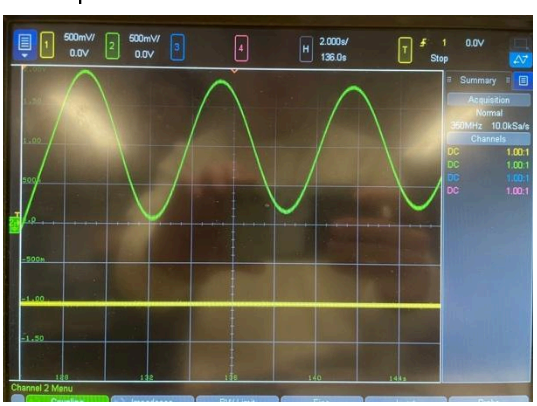
R3 controls the weight of the term of the first derivative of x(t). When R3 is shorted out (R3 = 0), the coefficient of the first-derivative term will be zero, and the system does not damp. It will oscillate forever.
The rise time of V1(t) is set to be 0 ns.

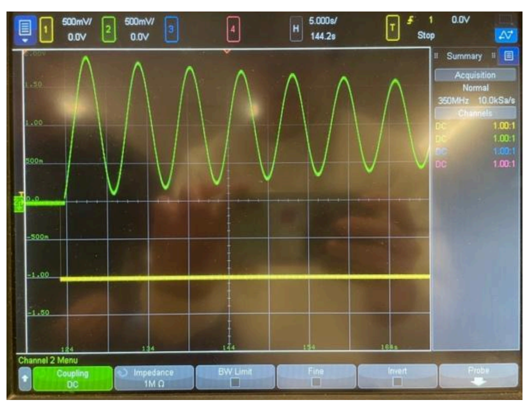


This is a cosine wave. x(t) = 1 - cos(t).
There is no exponential decay factor to dampen the response. It will keep oscillating.

Physical Experiment

We replaced R3 with a wire.





The amplitude of the first cycle was around 1V, which matched the simulated response. However, the response began to slightly decay afterward. Still, the sinusoidal oscillation was still visible after 40 seconds.

(c) Critical Damping

$$\ddot{x} + \frac{K_3}{\tau_1}\dot{x} + \frac{K_1}{\tau_1\tau_2}x = \frac{K_2}{\tau_1\tau_2}V_1$$

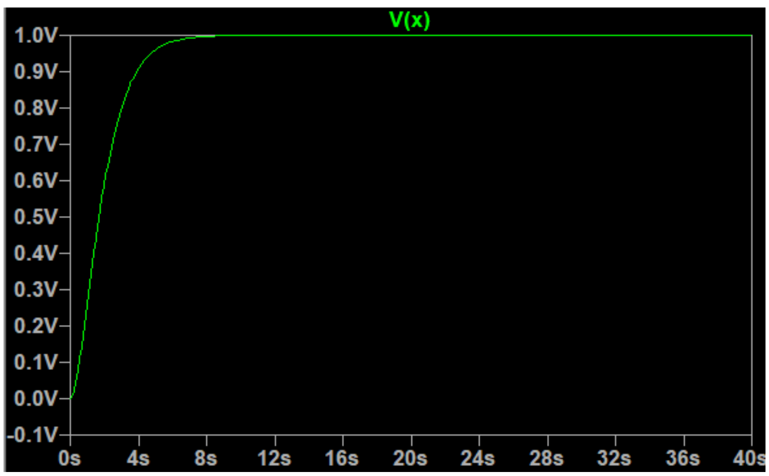
$2\zeta\omega_0 = \frac{K_3}{\tau_1}$. want $\zeta = 1$ for crit. damping.

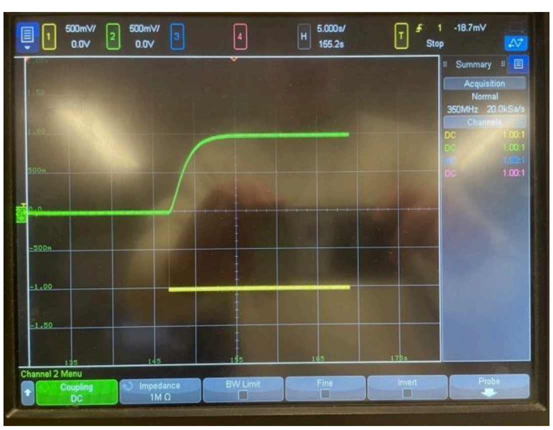
$\omega_0 = \frac{\sqrt{K_1}}{\tau}$ fixed to be 1, so $\frac{K_3}{\tau_1} = 2\cdot 1$

$$\frac{R_3R_5}{R_4R_8}\cdot\frac{1}{R_1C_1} = 2.$$

With all R fixed at 1M, all C fixed at 1μ, $R_3 = 2M\Omega$ for crit. damping

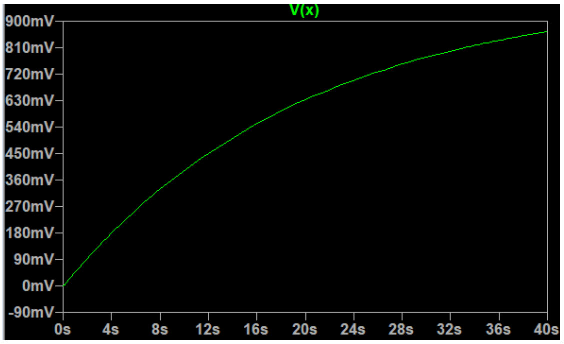
Critically damped, R3 = 2 Meg





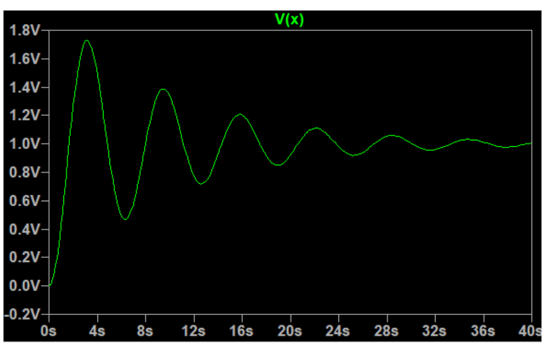
The results matched perfectly.

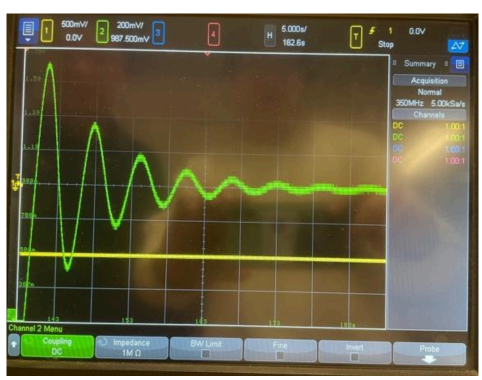
Overdamped, R3 = 20 Meg



Perfect match. This damping is much slower than the critical damping case.

Underdamped, R3 = 0.2 Meg





Again, perfect match. The step response oscillates.

(d) Higher Frequency Parameters

(d) $$\ddot{x} + \frac{K_3}{\tau_1}\dot{x} + \frac{K_1}{\tau_1 \tau_2}x = \frac{K_2}{\tau_1 \tau_2}V_1$$

$\omega_0^2 = \frac{K_1}{\tau^2} = 10000$. Fix K_1 and vary τ.

$0.01s = \tau = R_1 C_1 = R_2 C_2 = 100k\Omega \cdot 0.1\mu F \rightarrow R_1 = R_2 = 100k\Omega,\ C_1 = C_2 = 0.1\mu F$

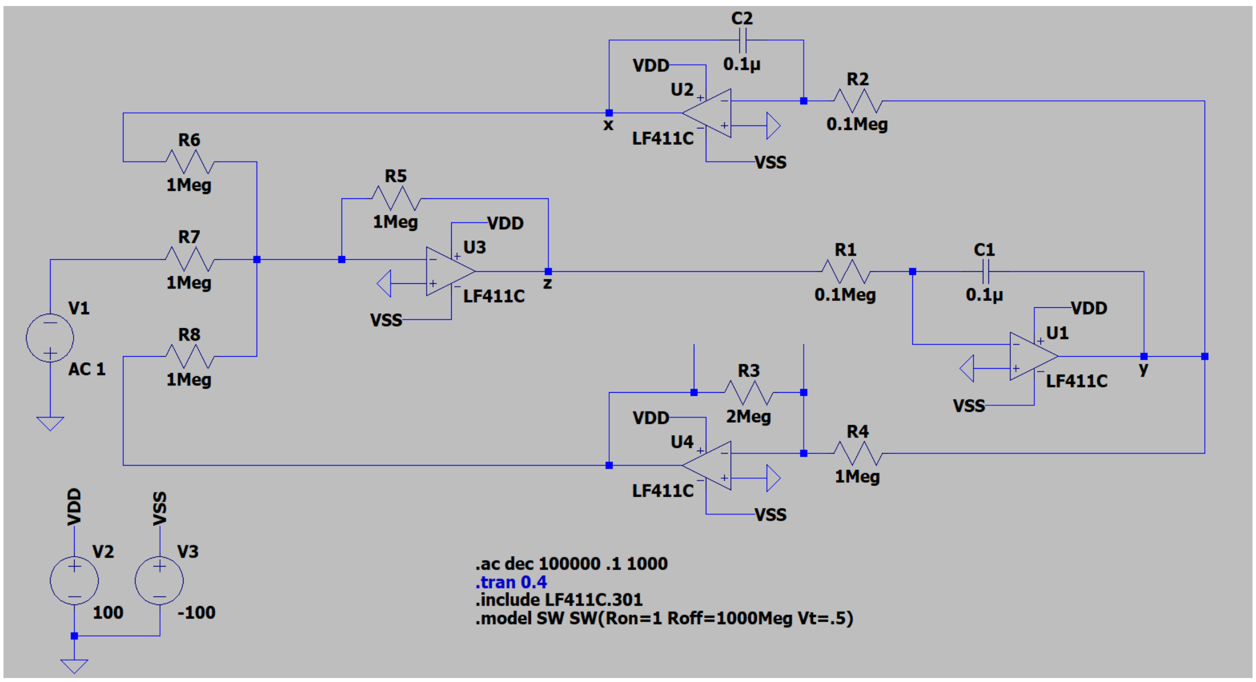
For critical damping, $\frac{K_3}{\tau_1} = 2\omega_0 = 200$

$$K_3 = \frac{R_3 R_5}{R_4 R_8} = 200\,s^{-1} \cdot 0.01s = 2$$

Make $R_3 = 2M\Omega$. Other params unchanged: $R_4, R_5, R_6, R_7, R_8 = 1M\Omega$

I only modified tau to prevent op amp saturation in any of the stages.

Design:

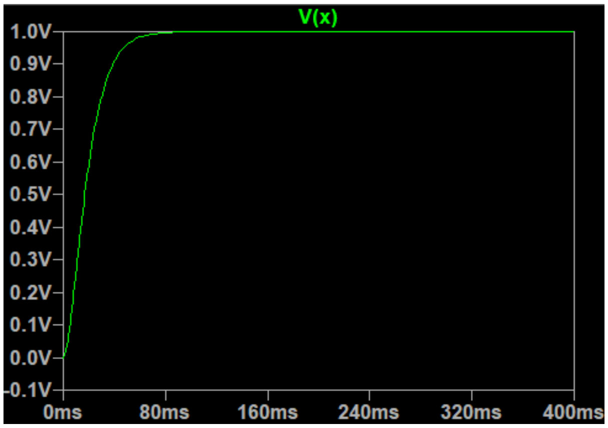


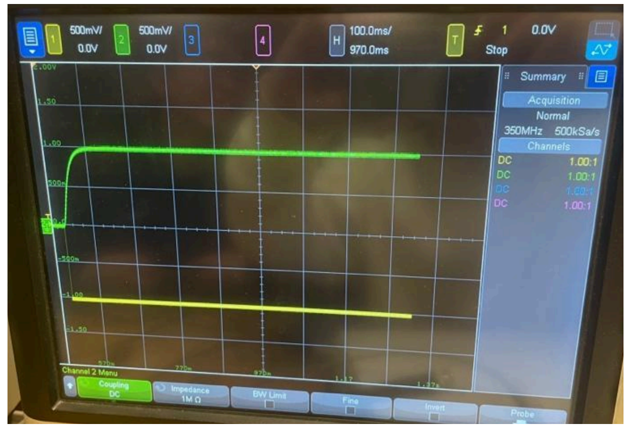
Since the frequency increases 100 times, I ran the transient simulation for 0.4 s instead of 40 s.

In the physical experiment, we only need to change R1, C1, R2, C2, and vary R3 accordingly.

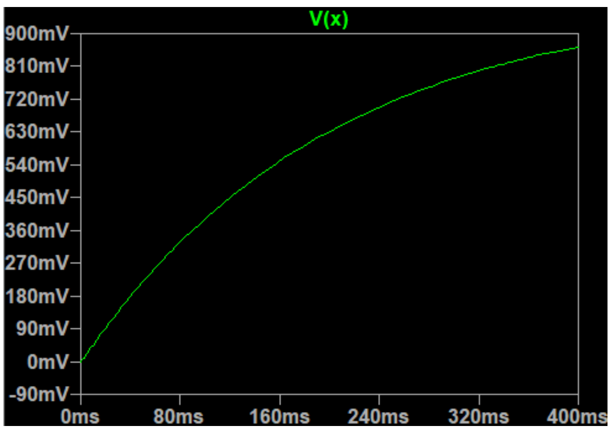
The 0.1u capacitor did not work, so we instead used the backup design R1 = R2 = 10k, and kept C1 = C2 = 1u. The products R1C1 and R2C2 are unaffected.

Critically damped, R3 = 2 Meg

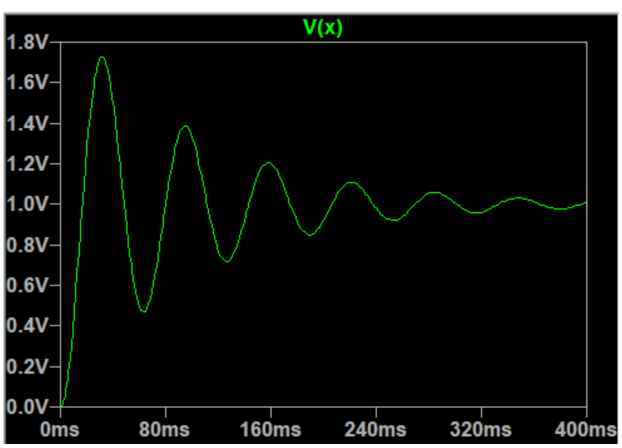




Overdamped, R3 = 20 Meg

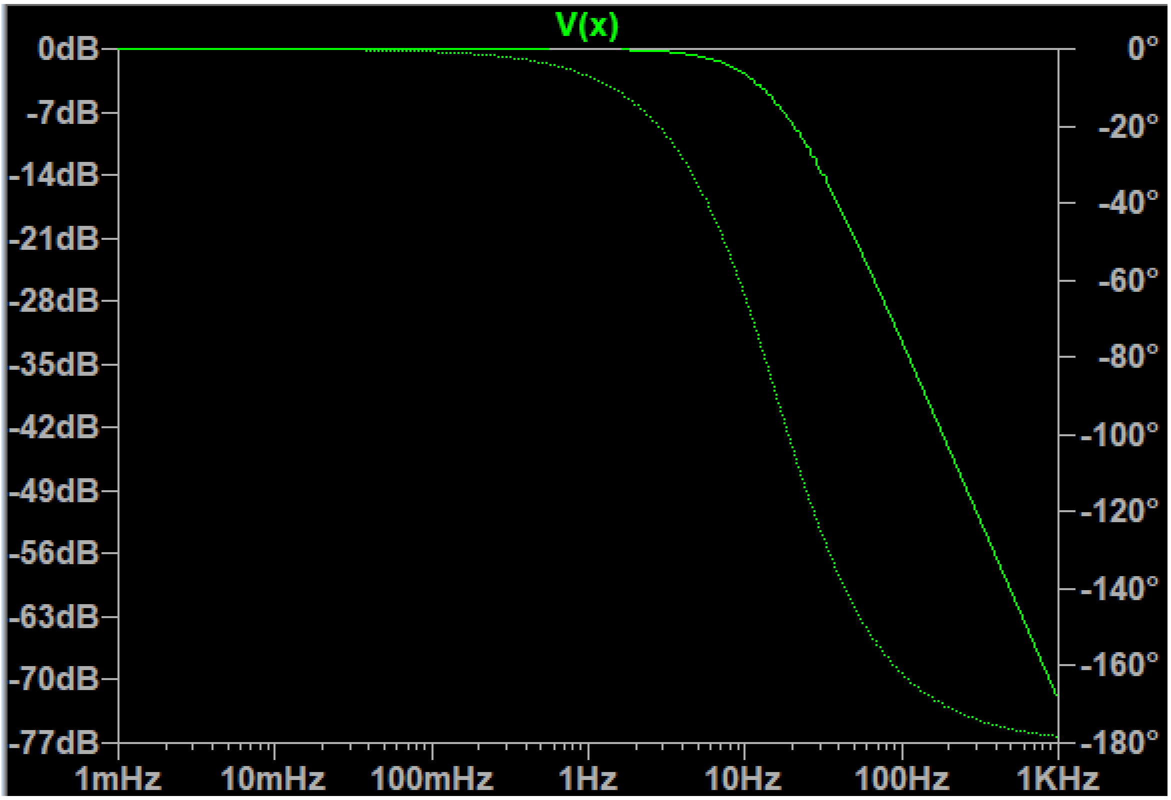


Underdamped, R3 = 0.2 Meg



All results matched perfectly with the simulation!

(e) AC Simulation



All parameters are left unchanged. R3 is set to 2 Meg for critical damping.

Calculated frequency response:

$$\ddot{x} + \frac{K_3}{\tau_1}\dot{x} + \frac{K_1}{\tau_1\tau_2}x = \frac{K_2}{\tau_1\tau_2}V_1$$

$$(j\omega)^2\underline{V_x} + \frac{K_3}{\tau}(j\omega)\underline{V_x} + \frac{K_1}{\tau^2}\underline{V_x} = \frac{K_2}{\tau^2}\underline{V_1} \qquad (\text{let } \tau_1 = \tau_2)$$

$$\underline{H}(j\omega) = \frac{\underline{V_x}}{\underline{V_1}} = \frac{K_2}{K_1 + (j\omega)K_3\tau + (j\omega)^2\tau^2}$$

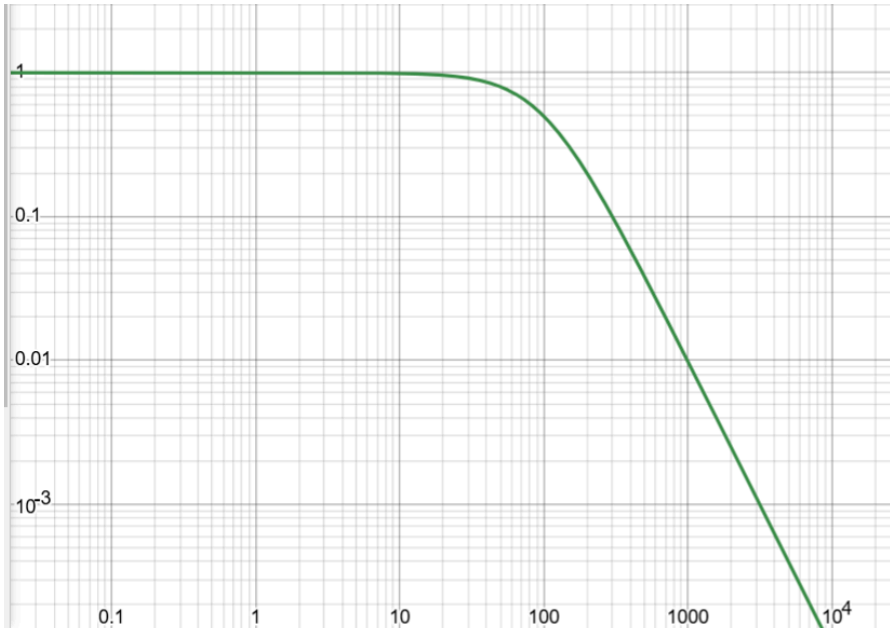
At critical damping, $K_1 = K_2 = 1,\ K_3 = 2,\ \tau = 0.01\,s = \frac{1}{\omega_0}$

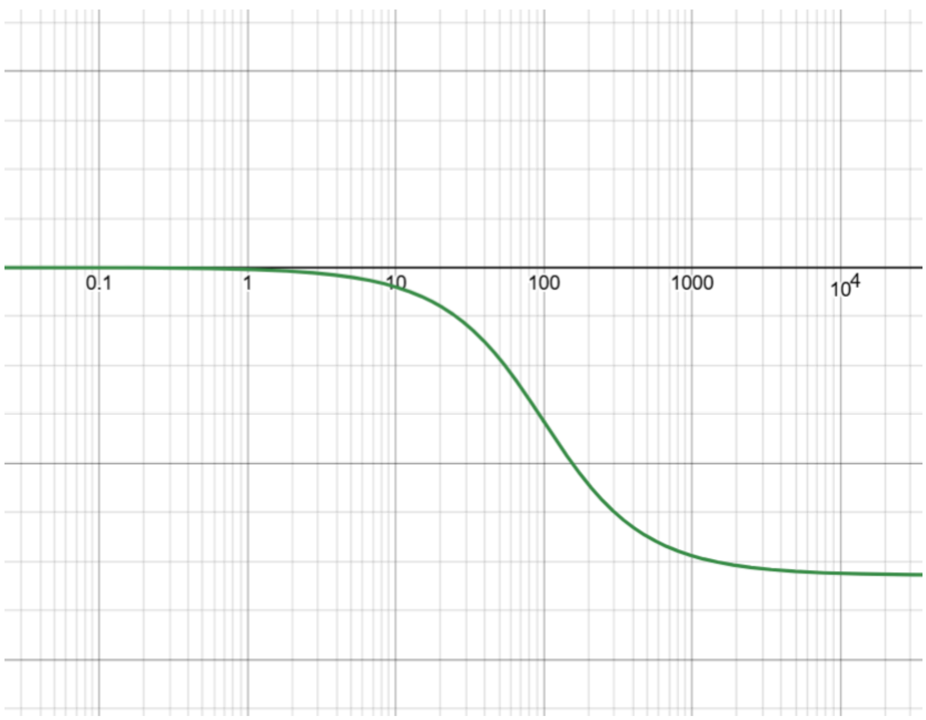
$$\underline{H_x}(j\omega) = \frac{1}{1 + 2(j\frac{\omega}{\omega_0}) + (j\frac{\omega}{\omega_0})^2}$$

$$M_x(\omega) = \frac{1}{\sqrt{(1-(\frac{\omega}{100})^2)^2 + (2\frac{\omega}{100})^2}}$$

$$\phi_x(\omega) = -\tan^{-1}\left(\frac{2\frac{\omega}{100}}{1-(\frac{\omega}{100})^2}\right) \qquad (-\pi \text{ when } \omega > 100)$$

Below are the gain and phase shift plots. For clarity, I left the gain in ratio, angle in radians, and frequency in rad/s. A simple unit conversion would show the equivalency.





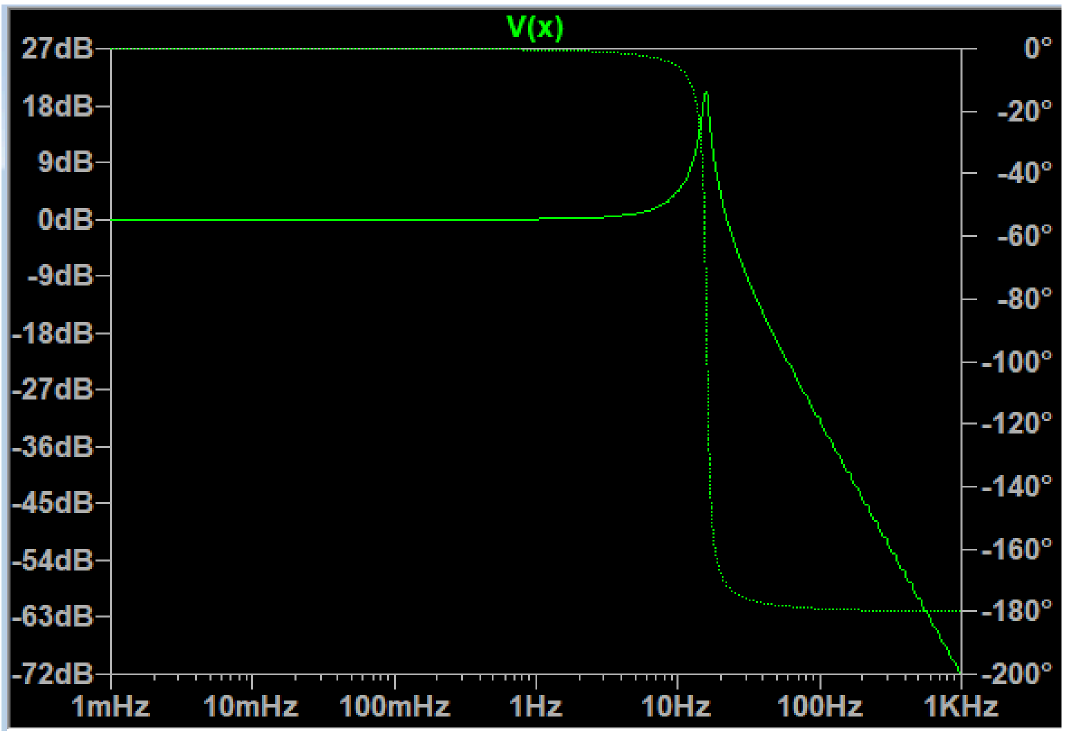
The phase shift went from 0 to -90 degrees.

(f) Quality Factor

(f) $Q = \frac{\sqrt{K_1}}{K_3} = 10$ (assume $\tau_1 = \tau_2$)

Let $\sqrt{K_1} = 1$, $K_3 = 0.1 \rightarrow R_3 = 0.1$ Meg

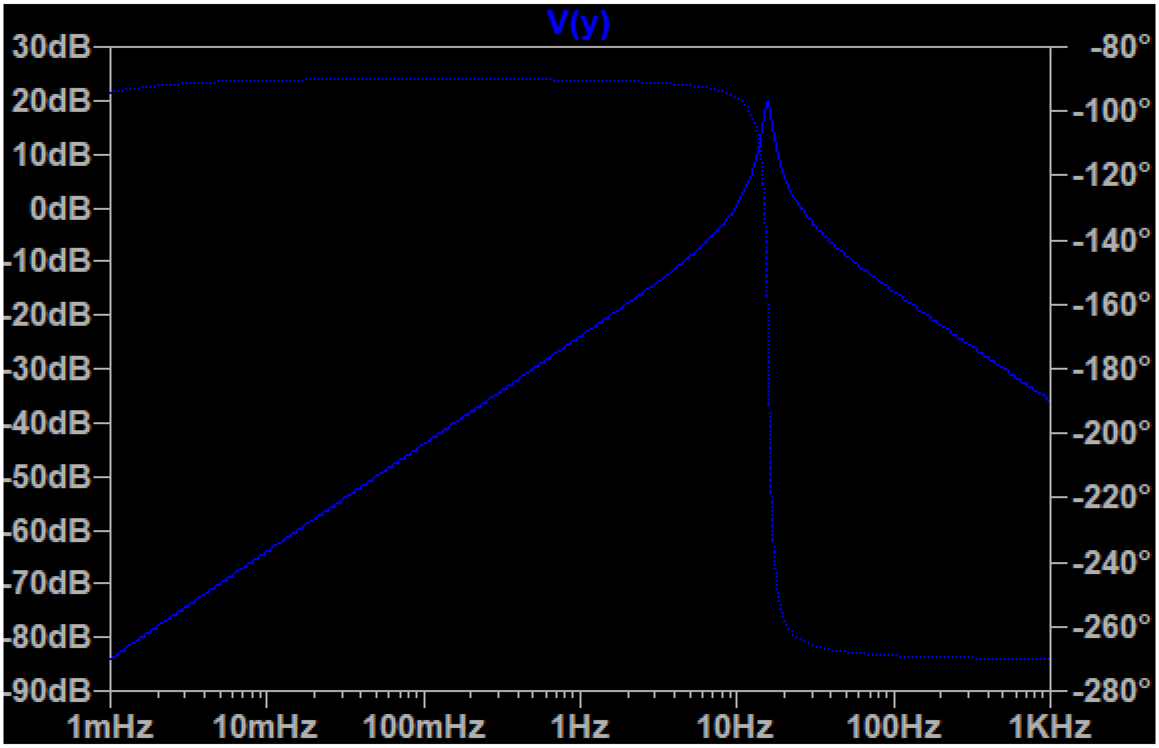
Make R3 = 0.1 Meg, the rest unchanged



There is a sharp peak of high gain at a resonant frequency of 100 rad/s. The phase shift is also sharper around resonance.

(g) First Derivative Frequency Response

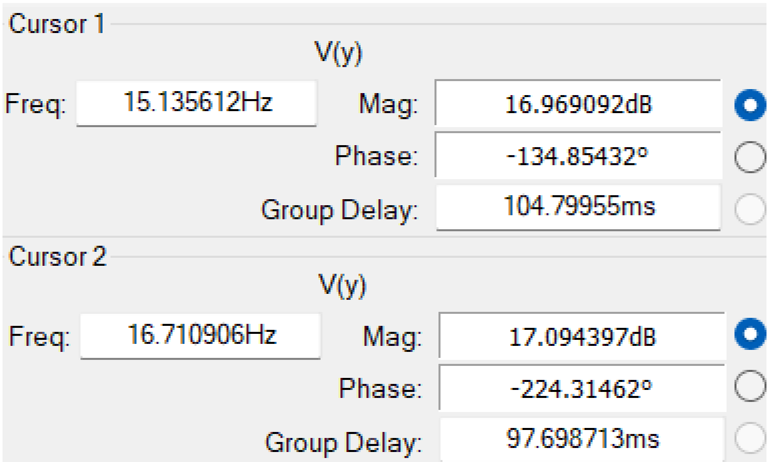
Let the output of U1 be vy(t). The value of R3 is retained from part (f): 0.1 Meg



This is a bandpass response.

(h) Behavior Analysis

The measured center frequency is 15.92 Hz. The peak gain is 20 dB. To measure bandwidth, I found frequencies where the gain is (20-3) dB. B = 16.71 - 15.14 = 1.57 Hz.



(g) Let V_Y be the output at U1. $\quad V_X(t) = -\frac{1}{R_2 C_2}\int V_Y(t)\,dt$

$$H_Y(j\omega) = \frac{V_Y}{V_1} = \frac{-K_2\tau(j\omega)}{K_1 + (j\omega)K_3\tau + (j\omega)^2\tau^2} \qquad -\tau_2(j\omega)V_X = V_Y$$

$$H_Y(j\omega) = \frac{-K_2(j\frac{\omega}{\omega_0})}{K_1 + K_3(j\frac{\omega}{100}) + (j\frac{\omega}{100})^2}$$

$\omega_r = \omega_0 = 100 \approx 15.92$ Hz

$$K_{max} = |H(\omega_r)| = \frac{\frac{100}{100}}{\sqrt{(1-\frac{100}{100})^2 + (0.1\cdot\frac{100}{100})^2}} = 10$$

From Desmos, $|H(\omega)| = \frac{K_{max}}{\sqrt{2}}$ at $\omega \approx 95.12$ and $\omega \approx 105.12$

$B = 105.12 - 95.12 = 10 \approx 1.59$ Hz

$Q = \frac{\omega_r}{B} = \frac{15.92\text{Hz}}{1.59\text{Hz}} = 10$ ✓

The calculated bandwidth matches the measured. The quality factor relation is also verified.

(i) Second Derivative Frequency Response

Let vz(t) be the output of op amp U3, which is proportional to the second derivative of vx(t). Let Hz(ω) be the ratio between Vz and the input V1. Hz is the third frequency response.

(i) $$K_2 V_{in} = \left(K_1 + (j\omega) K_3 \tau_2 + (j\omega)^2 \tau_1 \tau_2\right) \frac{1}{\tau_1 \tau_2 (j\omega)^2} V_z$$

$$H_z(j\omega) = \frac{V_z}{V_1} = \frac{K_2 \tau^2 (j\omega)^2}{K_1 + (j\omega) K_3 \tau + (j\omega)^2 \tau^2}$$